

理论力学 A

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作业 29 参考答案

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第六章哈密顿力学

19. 以向下为 x 轴的正方向，讨论落体运动. 设初始时刻的位移为 $x(0) = x_0$ ，动量为 $p(0) = p_0$.

- (1) 求出 $x_0 = x_0(t, x, p)$, $p_0 = p_0(t, x, p)$.
- (2) 证明 $x, p \rightarrow x_0, p_0$ 是正则变换.
- (3) 对此变换逐个判断四种常用生成函数是否存在；若存在，则求出各生成函数表达式和哈密顿函数 $K(t, x_0, p_0)$.
- (4) 分别求出初值 x_0, p_0 这两个守恒量所对应的准对称变换.

解答

(1) 系统的哈密顿函数为 (x 向下为正，势能 $V = -mgx$):

$$H = \frac{p^2}{2m} - mgx$$

正则方程为：

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = mg$$

积分得运动方程 (以 x_0, p_0 为初值):

$$p(t) = p_0 + mgt$$

$$x(t) = x_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2$$

反解上述方程组，用 t, x, p 表示 x_0, p_0 : 由第一式得：

$$p_0 = p - mgt$$

代入第二式：

$$x_0 = x - \frac{p - mgt}{m}t - \frac{1}{2}gt^2 = x - \frac{p}{m}t + gt^2 - \frac{1}{2}gt^2 = x - \frac{p}{m}t + \frac{1}{2}gt^2$$

综上：

$$\begin{cases} x_0 = x - \frac{p}{m}t + \frac{1}{2}gt^2 \\ p_0 = p - mgt \end{cases}$$

(2) 计算雅可比矩阵（或泊松括号）：

$$\{x_0, p_0\}_{x,p} = \frac{\partial x_0}{\partial x} \frac{\partial p_0}{\partial p} - \frac{\partial x_0}{\partial p} \frac{\partial p_0}{\partial x}$$

各偏导数为：

$$\begin{aligned} \frac{\partial x_0}{\partial x} &= 1, & \frac{\partial x_0}{\partial p} &= -\frac{t}{m} \\ \frac{\partial p_0}{\partial x} &= 0, & \frac{\partial p_0}{\partial p} &= 1 \end{aligned}$$

代入得：

$$\{x_0, p_0\}_{x,p} = 1 \cdot 1 - \left(-\frac{t}{m}\right) \cdot 0 = 1$$

由于泊松括号为 1，故该变换是正则变换。

(3) 由于 (x_0, p_0) 为常数，新哈密顿函数 K 应使得

$$\dot{x}_0 = \frac{\partial K}{\partial p_0} = 0, \dot{p}_0 = -\frac{\partial K}{\partial x_0} = 0.$$

故可取 $K = 0$. 变换关系需满足 $p\dot{x} - H = p_0\dot{x}_0 - K + \frac{dF}{dt}$ ，即

$$dF = pdx - p_0dx_0 + (K - H)dt = pdx - p_0dx_0 - Hdt.$$

(i) **第一类生成函数** $F_1(t, x, x_0)$ ：需满足 $p = \frac{\partial F_1}{\partial x}, p_0 = -\frac{\partial F_1}{\partial x_0}$. 由 (1) 中结果：

$$\begin{aligned} x_0 &= x - \frac{p}{m}t + \frac{1}{2}gt^2 \implies p = \frac{m(x - x_0)}{t} + \frac{1}{2}mgt \\ p_0 &= p - mgt = \frac{m(x - x_0)}{t} - \frac{1}{2}mgt \end{aligned}$$

由于可以解出 $p(x, x_0)$ 和 $p_0(x, x_0)$ ，且 $t \neq 0$ 时非奇异，**存在**. 积分 $\frac{\partial F_1}{\partial x} = \frac{m(x - x_0)}{t} + \frac{1}{2}mgt$ 得：

$$F_1 = \frac{m(x - x_0)^2}{2t} + \frac{1}{2}mgtx + f(x_0, t)$$

利用 $p_0 = -\frac{\partial F_1}{\partial x_0} = \frac{m(x - x_0)}{t} - \frac{\partial f}{\partial x_0}$ ，对比 p_0 表达式得 $\frac{\partial f}{\partial x_0} = \frac{1}{2}mgt \implies f = \frac{1}{2}mgtx_0 + g(t)$. 再由 $\frac{\partial F_1}{\partial t} + H = 0$ 确定 $g(t)$ ，最终可得（忽略常数项）：

$$F_1(t, x, x_0) = \frac{m(x - x_0)^2}{2t} + \frac{1}{2}mgt(x + x_0) - \frac{1}{24}mg^2t^3$$

(ii) 第二类生成函数 $F_2(t, x, p_0)$: 需满足 $p = \frac{\partial F_2}{\partial x}, x_0 = \frac{\partial F_2}{\partial p_0}$. 由 $p_0 = p - mgt \Rightarrow p = p_0 + mgt$. $x_0 = x - \frac{p_0 + mgt}{m}t + \frac{1}{2}gt^2 = x - \frac{p_0 t}{m} - \frac{1}{2}gt^2$. 可以表示为 (x, p_0) 的函数, **存在**. 积分 $\frac{\partial F_2}{\partial x} = p_0 + mgt \Rightarrow F_2 = (p_0 + mgt)x + f(p_0, t)$. 由 $x_0 = \frac{\partial F_2}{\partial p_0} = x + \frac{\partial f}{\partial p_0}$, 对比得 $\frac{\partial f}{\partial p_0} = -\frac{p_0 t}{m} - \frac{1}{2}gt^2$. 解得:

$$F_2(t, x, p_0) = (p_0 + mgt)x - \frac{p_0^2 t}{2m} - \frac{1}{2}gt^2 p_0 - \frac{1}{6}mg^2 t^3$$

(iii) 第三类生成函数 $F_3(t, p, x_0)$: 需满足 $x = -\frac{\partial F_3}{\partial p}, p_0 = -\frac{\partial F_3}{\partial x_0}$. 由 $p_0 = p - mgt, x = x_0 + \frac{p}{m}t - \frac{1}{2}gt^2$. 可以表示为 (p, x_0) 的函数, **存在**. 同理积分可得:

$$F_3(t, p, x_0) = (mgt - p)x_0 - \frac{p^2 t}{2m} + \frac{1}{2}gt^2 p - \frac{1}{6}mg^2 t^3$$

(iv) 第四类生成函数 $F_4(t, p, p_0)$: 需满足 $x = -\frac{\partial F_4}{\partial p}, x_0 = \frac{\partial F_4}{\partial p_0}$. 这就要求能从 p, p_0 反解出 x, x_0 . 但由 (1) 知 $p_0 = p - mgt$, 即 $p - p_0 = mgt$. 在固定时刻 t , p 和 p_0 不是独立的变量 (或者说雅可比行列式为 0), 无法反解出 x . 故 F_4 **不存在**.

(4) 设守恒量 I 为生成元 G , 变换为 $\delta x = \epsilon \frac{\partial G}{\partial p}, \delta p = -\epsilon \frac{\partial G}{\partial x}$.

(i) 对于 $p_0 = p - mgt$:

$$\delta x = \epsilon \frac{\partial p_0}{\partial p} = \epsilon, \quad \delta p = -\epsilon \frac{\partial p_0}{\partial x} = 0$$

对应变换: **空间平移** $x \rightarrow x + \epsilon, p \rightarrow p$. 验证准对称性:

$$\delta L = \delta \left(\frac{1}{2}m\dot{x}^2 + mgx \right) = mg\delta x = mg\epsilon = \frac{d}{dt}(mg\epsilon t).$$

(ii) 对于 $x_0 = x - \frac{pt}{m} + \frac{1}{2}gt^2$:

$$\delta x = \epsilon \frac{\partial x_0}{\partial p} = -\epsilon \frac{t}{m}, \quad \delta p = -\epsilon \frac{\partial x_0}{\partial x} = -\epsilon$$

对应变换: **一种与时间有关的参考系变换 (类伽利略变换)** $x \rightarrow x - \epsilon \frac{t}{m}, p \rightarrow p - \epsilon$. 验证准对称性:

$$\delta L = m\dot{x}\delta\dot{x} + mg\delta x = m\dot{x}\left(-\frac{\epsilon}{m}\right) + mg\left(-\frac{\epsilon t}{m}\right) = -\epsilon\dot{x} - \frac{\epsilon gt}{m} = \frac{d}{dt}\left(-\epsilon x - \frac{\epsilon gt^2}{2m}\right).$$

20. 分别用第一、三、四类生成函数作为主函数, (如果可行) 推导哈密顿雅可比方程.

解答

哈密顿-雅可比方程的核心思想是寻找生成函数 S , 使得新哈密顿量 $K = H + \frac{\partial S}{\partial t} = 0$, 从而新坐标和新动量均为常数. 通常 S 取为第二类生成函数 $S(q, P, t)$. 若选用其他类型:

1. **第一类生成函数** $S_1(q, Q, t)$: 关系式: $p = \frac{\partial S_1}{\partial q}, P = -\frac{\partial S_1}{\partial Q}$. 方程:

$$H\left(q, \frac{\partial S_1}{\partial q}, t\right) + \frac{\partial S_1}{\partial t} = 0$$

这是关于 S_1 的偏微分方程, 可行.

2. **第三类生成函数** $S_3(p, Q, t)$: 关系式: $q = -\frac{\partial S_3}{\partial p}, P = -\frac{\partial S_3}{\partial Q}$. 方程:

$$H\left(-\frac{\partial S_3}{\partial p}, p, t\right) + \frac{\partial S_3}{\partial t} = 0$$

这是关于 S_3 的偏微分方程, 可行.

3. **第四类生成函数** $S_4(p, P, t)$: 关系式: $q = -\frac{\partial S_4}{\partial p}, Q = \frac{\partial S_4}{\partial P}$. 方程:

$$H\left(-\frac{\partial S_4}{\partial p}, p, t\right) + \frac{\partial S_4}{\partial t} = 0$$

这是关于 S_4 的偏微分方程, 可行.

21. 质点在重力场中运动, 取水平方向为 x 轴, 垂直向上为 y 轴, 设初速度为 (v_1, v_2) . 利用哈密顿-雅可比方程求轨道方程.

解答

哈密顿函数 (y 向上):

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + mgy$$

令 $H = E = \alpha_1$. H-J 方程为:

$$\frac{1}{2m} \left[\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 \right] + mgy + \frac{\partial S}{\partial t} = 0$$

分离变量 $S = W(x, y) - \alpha_1 t$.

$$\frac{1}{2m} \left[\left(\frac{\partial W}{\partial x} \right)^2 + \left(\frac{\partial W}{\partial y} \right)^2 \right] + mgy = \alpha_1$$

设 $W(x, y) = W_x(x) + W_y(y)$, 分离常数为 α_x :

$$\left(\frac{dW_x}{dx}\right)^2 = \alpha_x^2 \implies W_x = \alpha_x x$$

$$\frac{1}{2m}\alpha_x^2 + \frac{1}{2m}\left(\frac{dW_y}{dy}\right)^2 + mgy = \alpha_1 \implies W_y = \int \sqrt{2m\alpha_1 - \alpha_x^2 - 2m^2gy} dy$$

主函数:

$$S = \alpha_x x + \int \sqrt{2m\alpha_1 - \alpha_x^2 - 2m^2gy} dy - \alpha_1 t$$

根据雅可比定理, 轨道方程由 $\frac{\partial S}{\partial \alpha_x} = \beta$ 给出 (β 为常数):

$$\frac{\partial S}{\partial \alpha_x} = x + \int \frac{-2\alpha_x}{2\sqrt{2m\alpha_1 - \alpha_x^2 - 2m^2gy}} dy = \beta$$

$$x - \alpha_x \int \frac{dy}{\sqrt{C - 2m^2gy}} = \beta \quad (C = 2m\alpha_1 - \alpha_x^2)$$

积分得:

$$x - \alpha_x \frac{-1}{m^2g} \sqrt{2m\alpha_1 - \alpha_x^2 - 2m^2gy} = \beta$$

代入初始条件确定常数: $t = 0$ 时 $x = 0, y = 0$. 初动量 $p_{x0} = mv_1, p_{y0} = mv_2$.
 $\alpha_x = p_x = mv_1$. $\alpha_1 = E = \frac{1}{2}m(v_1^2 + v_2^2)$. $\beta = 0 - mv_1 \frac{-1}{m^2g} \sqrt{2mE - m^2v_1^2} =$
 $\frac{v_1}{mg} \sqrt{m^2v_2^2} = \frac{v_1v_2}{g}$. 轨道方程为:

$$x - \frac{v_1v_2}{g} = -\frac{v_1}{mg} \sqrt{m^2v_2^2 - 2m^2gy}$$

$$x - \frac{v_1v_2}{g} = -\frac{v_1}{g} \sqrt{v_2^2 - 2gy}$$

两边平方整理:

$$\begin{aligned} \left(x - \frac{v_1v_2}{g}\right)^2 &= \frac{v_1^2}{g^2}(v_2^2 - 2gy) \\ x^2 - \frac{2v_1v_2}{g}x + \frac{v_1^2v_2^2}{g^2} &= \frac{v_1^2v_2^2}{g^2} - \frac{2v_1^2}{g}y \\ x^2 - \frac{2v_1v_2}{g}x &= -\frac{2v_1^2}{g}y \end{aligned}$$

解出 y :

$$y = \frac{v_2}{v_1}x - \frac{g}{2v_1^2}x^2$$

此即抛体运动的轨迹方程.