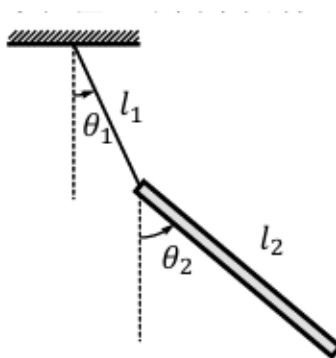


理论力学作业 08 参考答案

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以下两题选做其一：

14. 把一根质量为 m 的均匀细棒，用无弹性的轻绳（忽略质量）栓在天花板上。设细棒和绳子在垂直平面中摆动。写出系统的拉氏量和拉氏方程。



Solution

以图中的 θ_1 和 θ_2 为广义坐标，棒的质心为 $(l_1 \cos \theta_1 + \frac{1}{2}l_2 \cos \theta_2, l_1 \sin \theta_1 + \frac{1}{2}l_2 \sin \theta_2)$ ，棒的动能为质心动能加上棒绕质心转动的动能：

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}_c^2 + \frac{1}{2}I\dot{\theta}_2^2 \\ &= \frac{1}{2}m(l_1^2\dot{\theta}_1^2 + \frac{1}{4}l_2^2\dot{\theta}_2^2 + l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)) + \frac{1}{24}ml_2^2\dot{\theta}_2^2 \\ &= \frac{1}{2}ml_1^2\dot{\theta}_1^2 + \frac{1}{6}ml_2^2\dot{\theta}_2^2 + \frac{1}{2}ml_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \end{aligned}$$

$$V = -mg(l_1 \cos \theta_1 + \frac{l_2 \cos \theta_2}{2})$$

$$L = T - V$$

对 θ_1 有拉格朗日方程：

Solution

$$\begin{aligned}
 \frac{\partial L}{\partial \theta_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} &= -\frac{1}{2} m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m g l_1 \sin \theta_1 - \frac{d}{dt} \left(m l_1^2 \dot{\theta}_1 + \frac{1}{2} m l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right) \\
 &= -\frac{1}{2} m l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - m g l_1 \sin \theta_1 - m l_1^2 \ddot{\theta}_1 - \frac{1}{2} m l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\
 &\quad + \frac{1}{2} m l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\
 &= -\frac{1}{2} m l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - m g l_1 \sin \theta_1 - m l_1^2 \ddot{\theta}_1 - \frac{1}{2} m l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\
 &= 0
 \end{aligned}$$

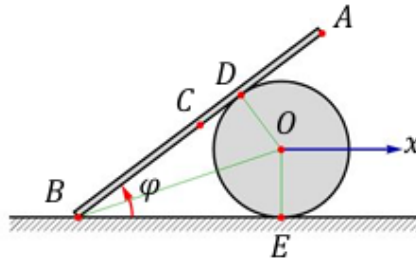
对 θ_2 有拉格朗日方程:

$$\begin{aligned}
 \frac{\partial L}{\partial \theta_2} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} &= \frac{1}{2} m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \frac{1}{2} m g l_2 \sin \theta_2 \\
 &\quad - \frac{d}{dt} \left(\frac{1}{3} m l_2^2 \dot{\theta}_2 + \frac{1}{2} m l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \right) \\
 &= \frac{1}{2} m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \frac{1}{2} m g l_2 \sin \theta_2 - \frac{1}{3} m l_2^2 \ddot{\theta}_2 - \frac{1}{2} m l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) \\
 &\quad + \frac{1}{2} m l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\
 &= \frac{1}{2} m l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - \frac{1}{2} m g l_2 \sin \theta_2 - \frac{1}{3} m l_2^2 \ddot{\theta}_2 - \frac{1}{2} m l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) \\
 &= 0
 \end{aligned}$$

综上, 系统拉氏方程为:

$$\begin{aligned}
 \frac{1}{2} l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g \sin \theta_1 + l_1 \ddot{\theta}_1 + \frac{1}{2} l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) &= 0 \\
 \frac{1}{2} l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - \frac{1}{2} g \sin \theta_2 - \frac{1}{3} l_2 \ddot{\theta}_2 - \frac{1}{2} l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) &= 0
 \end{aligned}$$

15. 半径为 a 、质量 M 的均匀圆盘, 与地面之间作纯滚动。长 l 、质量为 m 均匀细棒靠在圆盘上, 且棒身与圆盘的接触点处始终没有滑动; 另一端与光滑地面接触。以杆与地面的夹角 φ 为广义坐标, 写出系统的拉氏函数和运动方程。



Solution

以向右为正方向建立 X 轴，向上为正方向为 y 轴，并将 y 轴原点设为地面，使用图中 φ 作为广义坐标。

则各点坐标为: $O(x_o, a), B(x_B, 0), C(x_B + \frac{l}{2} \cos \varphi, \frac{l}{2} \sin \varphi), D(x_B + l_{BD} \cos \varphi, l_{BD} \sin \varphi)$

由几何关系得: $l_{BD} = x_o - x_B, (x_o - x_B) \tan \frac{\varphi}{2} = a$, 求导得:

$$\dot{x}_o - \dot{x}_B = -\frac{a}{2} \frac{1}{\sin^2 \frac{\varphi}{2}} \dot{\varphi}$$

设圆盘顺时针转动角度为 θ ，由纯滚动关系得:

$$\dot{x}_o = a\dot{\theta}$$

考虑棒身和圆盘接触点无滑动，即 D 对应棒上点的速度和 D 对应圆盘上点的速度在切向相同，棒上 D 点切向速度为:

$$v = \vec{v}_D \cdot (\cos \varphi, \sin \varphi) = (\dot{x}_D - l_{BD} \sin \varphi \dot{\varphi}, l_{BD} \sin \varphi \dot{\varphi}) \cdot (\cos \varphi, \sin \varphi) = \dot{x}_B \cos \varphi$$

圆盘上 D 点切向速度为:

$$v = a\dot{\theta} + \dot{x}_o \cos \varphi$$

二者相等，得

$$\dot{x}_B \cos \varphi = \dot{x}_o (\cos \varphi + 1)$$

与几何关系式 $\dot{x}_o - \dot{x}_B = -\frac{a}{2} \frac{1}{\sin^2 \frac{\varphi}{2}} \dot{\varphi}$ 联立可将 $\dot{x}_o$ 和 $\dot{x}_B$ 用广义坐标 φ 表示:

$$\dot{x}_o = \frac{a \cos \varphi}{2 \sin^2 \frac{\varphi}{2}} \dot{\varphi}$$

$$\dot{x}_B = a \cot^2 \frac{\varphi}{2} \dot{\varphi}$$

系统总动能包括圆盘质心动能、圆盘绕质心转动动能、棒质心动能、棒绕质心转动动能:

$$\begin{aligned} T &= \frac{1}{2} M \dot{x}_o^2 + \frac{1}{2} \cdot \frac{1}{2} M a^2 \dot{\theta}^2 + \frac{1}{2} m (\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{24} m l^2 \dot{\varphi}^2 \\ &= \frac{3}{4} M \left(\frac{a \cos \varphi}{2 \sin^2 \frac{\varphi}{2}} \dot{\varphi} \right)^2 + \frac{1}{24} m l^2 \dot{\varphi}^2 + \frac{1}{2} m \left[(\dot{x}_B - \frac{l}{2} \sin \varphi \dot{\varphi})^2 + (\frac{l}{2} \cos \varphi \dot{\varphi})^2 \right] \\ &= \frac{3}{4} M \left(\frac{a \cos \varphi}{2 \sin^2 \frac{\varphi}{2}} \dot{\varphi} \right)^2 + \frac{1}{24} m l^2 \dot{\varphi}^2 + \frac{1}{2} m \left[a^2 \cot^4 \frac{\varphi}{2} - a l \cot^2 \frac{\varphi}{2} \sin \varphi + \frac{l^2}{4} \right] \dot{\varphi}^2 \\ &= \dot{\varphi}^2 \left(\frac{3}{16} M a^2 \frac{\cos^2 \varphi}{\sin^4 \frac{\varphi}{2}} + \frac{1}{6} m l^2 + \frac{1}{2} m a^2 \cot^4 \frac{\varphi}{2} - \frac{1}{2} m a l \cot^2 \frac{\varphi}{2} \sin \varphi \right) \\ &\equiv A(\varphi) \dot{\varphi}^2 \end{aligned}$$

Solution

系统势能只有棒受到的重力势能：

$$V = \frac{l}{2}mg \sin \varphi$$

系统拉氏函数为：

$$L = \dot{\varphi}^2 \left(\frac{3}{16}Ma^2 \frac{\cos^2 \varphi}{\sin^4 \frac{\varphi}{2}} + \frac{1}{6}ml^2 + \frac{1}{2}ma^2 \cot^4 \frac{\varphi}{2} - \frac{1}{2}mal \cot^2 \frac{\varphi}{2} \sin \varphi \right) - \frac{l}{2}mg \sin \varphi$$

$$\frac{\partial L}{\partial \varphi} = -\frac{1}{2}mg \cos \varphi + \dot{\varphi}^2 \frac{\partial A}{\partial \varphi}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} = 2A\ddot{\varphi} + 2\dot{\varphi}^2 \frac{\partial A}{\partial \varphi}$$

计算 $\frac{\partial A}{\partial \varphi}$ ：

$$\begin{aligned} \frac{\partial A}{\partial \varphi} = & -\frac{3Ma^2}{8} \cdot \frac{\cos \varphi \sin \varphi \sin \frac{\varphi}{2} + \cos^2 \varphi \cos \frac{\varphi}{2}}{\sin^5 \frac{\varphi}{2}} - ma^2 \cot^3 \frac{\varphi}{2} \csc^2 \frac{\varphi}{2} \\ & - \frac{mal}{2} \left(-\cot \frac{\varphi}{2} \csc^2 \frac{\varphi}{2} \sin \varphi + \cot^2 \frac{\varphi}{2} \cos \varphi \right) \end{aligned}$$

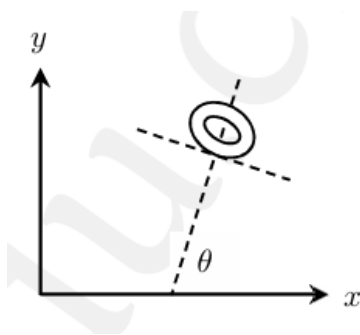
故最终运动方程为：

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 2A\ddot{\varphi} + \dot{\varphi}^2 \frac{\partial A}{\partial \varphi} + \frac{1}{2}mg \cos \varphi = 0$$

代入 A 和 $\frac{\partial A}{\partial \varphi}$ 表达式：

$$\begin{aligned} & 2 \left[\frac{3Ma^2 \cos^2 \varphi}{16 \sin^4 \frac{\varphi}{2}} + \frac{ml^2}{6} + \frac{ma^2 \cot^4 \frac{\varphi}{2}}{2} - \frac{mal \cot^2 \frac{\varphi}{2} \sin \varphi}{2} \right] \ddot{\varphi} \\ & + \left[-\frac{3Ma^2}{8} \cdot \frac{\cos \varphi \sin \varphi \sin \frac{\varphi}{2} + \cos^2 \varphi \cos \frac{\varphi}{2}}{\sin^5 \frac{\varphi}{2}} - ma^2 \cot^3 \frac{\varphi}{2} \csc^2 \frac{\varphi}{2} \right. \\ & \left. - \frac{mal}{2} \left(-\cot \frac{\varphi}{2} \csc^2 \frac{\varphi}{2} \sin \varphi + \cot^2 \frac{\varphi}{2} \cos \varphi \right) \right] \dot{\varphi}^2 + \frac{mgl \cos \varphi}{2} \\ & = 0 \end{aligned}$$

16. 通过引入拉氏乘子，写出讲稿例题直立滚轮问题的运动方程。



Solution

取广义坐标为轮心的位置 (x, y) , 轮轴与 x 轴夹角 θ , 轮子绕轴转过的角度 φ 。
轮子前进方向为 $\theta - \frac{\pi}{2}$, 由无滑滚动条件得:

$$\dot{x} - R \cos(\theta - \frac{\pi}{2})\dot{\varphi} = 0, \quad \dot{y} + R \sin(\theta - \frac{\pi}{2})\dot{\varphi} = 0$$

写为变分约束:

$$\delta x - R \sin \theta \delta \varphi = 0, \quad \delta y + R \cos \theta \delta \varphi = 0$$

系统动能包括轮心的动能, 轮子绕轴自转的转动动能, 和轮子绕竖直轴的转动动能。
故系统的拉氏函数为:

$$L = T = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\varphi}^2 + \frac{1}{2}I_2\dot{\theta}^2$$

对于变分约束方程 $c_{\alpha k}\delta q_k = 0$, 带乘子的拉氏方程为:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \lambda_{\alpha} c_{\alpha k}$$

本题中带乘子的拉氏方程为:

$$M\ddot{x} = \lambda_1$$

$$M\ddot{y} = \lambda_2$$

$$I_1\ddot{\varphi} = -R \sin \theta \lambda_1 + R \cos \theta \lambda_2$$

$$I_2\ddot{\theta} = 0$$

由第四个方程知

$$\theta = \omega_0 t + \theta_0$$

对两个约束方程求导得:

$$\ddot{x} = R \sin \theta \ddot{\varphi} + \omega_0 R \cos \theta \dot{\varphi}$$

$$\ddot{y} = -R \cos \theta \ddot{\varphi} + \omega_0 R \sin \theta \dot{\varphi}$$

Solution

代入拉氏方程得，

$$\lambda_1 = M(R \sin \theta \ddot{\varphi} + \omega_0 R \cos \theta \dot{\varphi})$$

$$\lambda_2 = M(-R \cos \theta \ddot{\varphi} + \omega_0 R \sin \theta \dot{\varphi})$$

$$(I_1 + MR^2)\ddot{\varphi} = 0$$

即 $\varphi = \omega_1 t + \varphi_0$

最后对两个约束方程积分，得

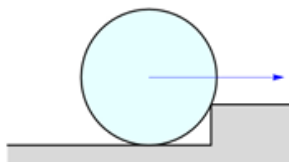
$$x = \int dx = \int R \sin \theta d\varphi = R \int \sin(\omega_0 t + \theta_0) \omega_1 dt = -\frac{\omega_1}{\omega_0} R \cos(\omega_0 t + \theta_0) + c_1$$

$$y = \int dy = -\int R \cos \theta d\varphi = -\int R \cos(\omega_0 t + \theta_0) \omega_1 dt = -\frac{\omega_1}{\omega_0} R \sin(\omega_0 t + \theta_0) + c_2$$

系统做圆周运动

17. 半径为 r 的圆盘，以速度 v 从左向右无滑滚动，直至碰上高 h 的台阶。设碰撞是弹性的，且 $r > h$ 。

- (1) 如果台阶边缘完全光滑，与圆盘边缘没有摩擦，求圆盘碰撞后的状态。
- (2) 若碰撞在中台阶边缘与圆盘有切向滑动，摩擦系数是 μ ，求碰撞后的状态。



Solution

(1)

记碰撞时圆盘质心与碰撞点连线与竖直线夹角为 α ，满足 $\cos \alpha = \frac{r-h}{r}$ 。

建立直角坐标系，圆盘质心位置为 (x, y) ，碰撞前速度为 $\vec{v} = v\hat{x}$ ，沿着作用力方向的速度为 $(v \sin^2 \alpha, -v \sin \alpha \cos \alpha)$ ，垂直作用力方向的速度为 $(v \cos^2 \alpha, v \cos \alpha \sin \alpha)$ 。

由纯滚动条件得 $v = r\dot{\theta}$ ，其中 θ 为圆盘转动角度。

圆盘边缘没有摩擦的碰撞，碰撞时台阶支持力作用于圆盘质心，圆盘不受力矩作用。由于是弹性碰撞，沿着作用力方向的速度反向，垂直于作用力方向速度不变。

故碰撞后，圆盘自转角速度不变仍为 $\dot{\theta} = \frac{v}{r}$ ，垂直作用力方向的速度仍为 $(v \cos^2 \alpha, v \cos \alpha \sin \alpha)$ ，沿着作用力方向的速度反向为 $(-v \sin^2 \alpha, v \sin \alpha \cos \alpha)$

Solution

因此碰撞后速度为

$$\vec{v} = (v \cos 2\alpha, v \sin 2\alpha)$$

(2)

碰撞时圆盘共受到两部分冲量，一部分为支持力对应的冲量记为 I_1 ，另一部分为摩擦力对应的冲量记为 I_2 。记碰撞点对应圆盘上的点为 D，设 θ 为 D 对应的和竖直方向夹角（以顺时针转动为正方向），则碰撞时 $\theta = -\alpha$ 。

D 的坐标为：

$$(x - r \sin \theta, y - r \cos \theta)$$

由于圆盘发生碰撞时受到的支持力与（1）中相同，故：

$$\vec{I}_1 = \vec{N} \delta(t) = m(v \cos 2\alpha - v, v \sin 2\alpha) = 2mv \sin \alpha (-\sin \alpha, \cos \alpha)$$

I_2 方向应沿着碰撞点切线方向

$$\vec{I}_2 = \mu N \delta(t) \hat{N}_\perp = \mu I_1 (\cos \alpha, \sin \alpha) = 2\mu m v \sin \alpha (\cos \alpha, \sin \alpha)$$

冲量在广义坐标的投影 $I_\alpha = \frac{\partial \vec{r}}{\partial q_\alpha} \cdot \vec{I}$ 为：

$$I_{1x} = -2mv \sin^2 \alpha, \quad I_{1y} = 2mv \sin \alpha \cos \alpha$$

$$I_{1\theta} = 2mvr \sin \alpha \sin (\alpha + \theta)$$

$$I_{2x} = 2\mu m v \sin \alpha \cos \alpha, \quad I_{2y} = 2\mu m v \sin^2 \alpha$$

$$I_{2\theta} = -2\mu mvr \sin \alpha \cos (\alpha + \theta)$$

系统的动能为：

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}mr^2\dot{\theta}^2$$

根据 $\left. \frac{\partial T}{\partial \dot{q}_\alpha} \right|_{0-}^{0+} = I_{1\alpha} + I_{2\alpha}$

$$m\dot{x} \Big|_{0-}^{0+} = 2mv \sin \alpha (\mu \cos \alpha - \sin \alpha)$$

$$m\dot{y} \Big|_{0-}^{0+} = 2mv \sin \alpha (\cos \alpha + \mu \sin \alpha)$$

$$\frac{1}{2}mr^2\dot{\theta} \Big|_{0-}^{0+} = 2mvr \sin \alpha (\sin (\theta + \alpha) - \mu \cos (\theta + \alpha))$$

Solution

又 $\dot{x} = v, \dot{y} = 0, \dot{\theta} = \frac{v}{r}$, 并代入碰撞时 $\theta = -\alpha$, 得系统碰撞后状态为

$$\dot{x}(0+) = \mu v \sin 2\alpha + v \cos 2\alpha$$

$$\dot{y}(0+) = 2v \sin \alpha (\cos \alpha + \mu \sin \alpha)$$

$$\dot{\theta}(0+) = \frac{v}{r}(1 - 4\mu \sin \alpha)$$