

理论力学作业 02 参考答案

助教：张积翔、于洪飞

6. 使用张量符号进行计算、化简

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = ?$$

$$\nabla \times (\nabla \times \vec{a}) = ?$$

$$\nabla \cdot (\psi \vec{a}) = ?$$

$$\nabla \times (\psi \vec{a}) = ?$$

Solution

$$\begin{aligned}\vec{a} \times (\vec{b} \times \vec{c}) &= \hat{e}_i \varepsilon_{ijk} a_j \varepsilon_{kmn} b_m c_n \\ &= \hat{e}_i a_j b_m c_n (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \\ &= \hat{e}_i b_i a_j c_j - c_i a_j b_j \\ &= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} = 0\end{aligned}$$

$$\begin{aligned}\nabla \times (\nabla \times \vec{a}) &= \hat{e}_i \varepsilon_{ijk} \partial_j (\varepsilon_{kmn} \partial_m a_n) \\ &= \hat{e}_i \varepsilon_{ijk} \varepsilon_{kmn} \partial_j \partial_m a_n \\ &= \hat{e}_i (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \partial_j \partial_m a_n \\ &= \hat{e}_i (\partial_i \partial_n a_n - \partial_j \partial_j a_i) \\ &= \nabla (\nabla \cdot \vec{a}) - \nabla^2 \vec{a}, \quad \text{where } \nabla^2 \equiv \nabla \cdot \nabla\end{aligned}$$

$$\begin{aligned}\nabla \cdot (\psi \vec{a}) &= \partial_i (\psi a_i) \\ &= (\partial_i \psi) a_i + \psi (\partial_i a_i) \\ &= (\nabla \psi) \cdot \vec{a} + \psi (\nabla \cdot \vec{a})\end{aligned}$$

$$\begin{aligned}\nabla \times (\psi \vec{a}) &= \hat{e}_i \varepsilon_{ijk} \partial_j (\psi a_k) \\ &= \hat{e}_i \varepsilon_{ijk} [(\partial_j \psi) a_k + \psi (\partial_j a_k)] \\ &= (\nabla \psi) \times \vec{a} + \psi (\nabla \times \vec{a})\end{aligned}$$

7. 矢量场 $\vec{B}$ 分别取成

(1) 均匀磁场

$$\vec{B} = -\frac{1}{2}\vec{B}^{(0)}$$

(2) 均匀梯度磁场

$$B_j = -\frac{1}{3}B_{jk}^{(1)}x_k$$

(3) 二次多项式磁场

$$B_j = -\frac{1}{4}B_{jkl}^{(2)}x_kx_l$$

三种形式。表达式中的系数 $B_j^{(0)}, B_{jk}^{(1)}, B_{jkl}^{(2)}$ 都是常数。为了满足无源条件

$$\nabla \cdot \vec{B} = 0$$

这些系数还满足

$$B_{jj}^{(1)} = 0, \quad B_{jj}^{(2)} = 0, \quad B_{jkl}^{(2)} = 0$$

对这三种情况, 请分别计算并化简

$$\nabla \times (\vec{x} \times \vec{B})$$

根据上述结果, 三种磁场的矢量势 $\vec{A}$ 分别是什么?

Solution

$$\begin{aligned} \left[\nabla \times (\vec{x} \times \vec{B}) \right]_i &= \varepsilon_{ijk} \partial_j (\varepsilon_{kmn} x_m B_n) \\ &= \varepsilon_{ijk} \varepsilon_{kmn} \partial_j (x_m B_n) \\ &= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \partial_j (x_m B_n) \\ &= \partial_j (x_i B_j) - \partial_j (x_j B_i) \\ &= B_i + x_i \partial_j B_j - 3B_i - x_j \partial_j B_i \\ &= -2B_i + x_i (\nabla \cdot \vec{B}) - (\vec{x} \cdot \nabla) B_i \end{aligned}$$

由于 $\nabla \cdot \vec{B} = 0$, 所以:

$$\nabla \times (\vec{x} \times \vec{B}) = -2\vec{B} - (\vec{x} \cdot \nabla) \vec{B}$$

对于三种磁场情况:

均匀磁场 $\vec{B} = -\frac{1}{2}\vec{B}^{(0)}$:

$$\begin{aligned} \nabla \times (\vec{x} \times \vec{B}) &= -2\vec{B} = \vec{B}^{(0)} \\ \vec{A} &= -\frac{1}{2}\vec{x} \times \vec{B} \end{aligned}$$

Solution

均匀梯度磁场 $B_j = -\frac{1}{3}B_{jk}^{(1)}x_k$:

$$\begin{aligned}\nabla \times (\vec{x} \times \vec{B}) &= -2\vec{B} + \frac{1}{3}\hat{e}_j x_i \partial_i (B_{jk}^{(0)} x_k) \\ &= -2\vec{B} + \frac{1}{3}\hat{e}_j x_i B_{ji}^{(1)} \\ &= -3\vec{B}\end{aligned}$$

$$\vec{A} = -\frac{1}{3}(\vec{x} \times \vec{B})$$

二次多项式磁场 $B_j = -\frac{1}{4}B_{jkl}^{(2)}x_k x_l$:

$$\begin{aligned}\nabla \times (\vec{x} \times \vec{B}) &= -2\vec{B} + \frac{1}{4}\hat{e}_j x_i \partial_i B_{jkl}^{(2)} x_k x_l \\ &= -4\vec{B} \\ A_i &= -\frac{1}{4}(\vec{x} \times \vec{B})\end{aligned}$$

8. 小船被水冲走后, 用缆绳拉回岸边。假定水流平行于河岸, 速率为 c_1 ; 拉绳的速率为 c_2 , 这两个速率均保持不变。

以水流方向为极轴, 拉绳者为原点, 建立平面极坐标系, 并在此坐标系求出小船的轨迹曲线。

Solution

在极坐标系中, 小船沿着半径方向的速度为 $v_r = -c_2$, 垂直半径方向的速度完全由水速提供, 故:

$$v_\theta = -c_1 \sin\theta$$

即 $\vec{v} = -c_2 \hat{r} - c_1 \sin\theta \hat{\theta}$, 又 $\vec{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$

$$\dot{r} = -c_2, \quad r \dot{\theta} = -c_1 \sin\theta$$

两式相除得:

$$\frac{dr}{r} = \frac{c_2}{c_1 \sin\theta} d\theta$$

解微分方程得小船运动轨迹为:

$$r = C \left(\tan \frac{\theta}{2} \right)^{\frac{c_2}{c_1}}$$

以下两题是额外的练习，不需要提交给助教，不计入作业成绩：

9. R 是转动矩阵，证明 $\vec{a} \times \vec{b}$ 像矢量一样转动：

$$R(\vec{a} \times \vec{b}) = (R\vec{a}) \times (R\vec{b})$$

Solution

$$\begin{aligned}(R\vec{a}) \times (R\vec{b}) &= \hat{e}_k \varepsilon_{kil} (R_{ij} a_j) (R_{lm} a_m) \\ &= \hat{e}_k \varepsilon_{kil} R_{ij} R_{lm} a_j a_m\end{aligned}$$

$$R(\vec{a} \times \vec{b}) = \hat{e}_k \varepsilon_{ijm} R_{ki} a_j a_m$$

由 $\vec{a}$ 和 $\vec{b}$ 的任意性，若要证题中等式成立，即证明：

$$\varepsilon_{kil} R_{ij} R_{lm} = \varepsilon_{ijm} R_{ki}$$

利用 $R^T R = I$,

$$\begin{aligned}\varepsilon_{kil} R_{ij} R_{lm} &= \varepsilon_{kil} R_{ij} R_{lm} \delta_{rk} \\ &= \varepsilon_{ril} R_{ij} R_{lm} R_{rn} R_{kn} \\ &= \varepsilon_{njm} \det(R) R_{kn} \\ &= \varepsilon_{ijm} R_{ki}\end{aligned}$$

题中等式得证

10. 设

$$\phi(\vec{x}) = -\vec{x} \cdot \int_0^1 \vec{E}(\lambda \vec{x}) d\lambda, \quad \vec{A}(\vec{x}) = -\int_0^1 \lambda \vec{x} \times \vec{B}(\lambda \vec{x}) d\lambda$$

其中积分项中的矢量场满足

$$\nabla \times \vec{E} = \vec{0}, \quad \nabla \cdot \vec{B} = 0$$

求 $-\nabla \phi, \nabla \times \vec{A}$ 。

提示：利用分部积分化简。这题给出了任意电场 $\vec{E}(\vec{x})$ 的标量势函数，以及任意磁场 $\vec{B}(\vec{x})$ 的矢量势函数。

Solution

计算 $-\nabla\phi$:

$$\begin{aligned} -\nabla\phi &= \nabla \left(\vec{x} \cdot \int_0^1 \vec{E}(\lambda\vec{x}) d\lambda \right) \\ &= \hat{e}_i \partial_i \left(x_j \int_0^1 E_j(\lambda\vec{x}) d\lambda \right) \\ &= \hat{e}_i \int_0^1 E_i(\lambda\vec{x}) d\lambda + \hat{e}_i x_j \int_0^1 \partial_i E_j(\lambda\vec{x}) d\lambda \end{aligned}$$

将第一项分部积分:

$$\begin{aligned} E_i(\lambda\vec{x}) &= \frac{d(\lambda E_i)}{d\lambda} - \lambda \frac{dE_i}{d\lambda} \\ \frac{dE_i}{d\lambda} &= \frac{\partial E_i(\lambda\vec{x})}{\partial(\lambda x_j)} \frac{d(\lambda x_j)}{d\lambda} = \frac{x_j}{\lambda} \partial_j E_i \end{aligned}$$

故:

$$\begin{aligned} -\nabla\phi &= \hat{e}_i \int_0^1 \left[\frac{d(\lambda E_i)}{d\lambda} - \lambda \frac{dE_i}{d\lambda} \right] d\lambda + \hat{e}_i x_j \int_0^1 \lambda \partial_i E_j(\lambda\vec{x}) d\lambda \\ &= \hat{e}_i \left[\lambda E_i(\lambda\vec{x}) \Big|_0^1 + \int_0^1 (x_j \partial_i E_j(\lambda\vec{x}) - x_j \partial_j E_i(\lambda\vec{x})) d\lambda \right], \nabla \times \vec{E} = 0 \\ &= \vec{E}(\vec{x}) \end{aligned}$$

计算 $\nabla \times \vec{A}$:

$$\begin{aligned} \nabla \times \vec{A} &= -\nabla \times \left(\int_0^1 \lambda \vec{x} \times \vec{B}(\lambda\vec{x}) d\lambda \right) \\ &= -\hat{e}_i \varepsilon_{ijk} \partial_j \int_0^1 \lambda \varepsilon_{kmn} x_m B_n(\lambda\vec{x}) d\lambda \\ &= -\hat{e}_i (\delta_{mi} \delta_{jn} - \delta_{in} \delta_{jm}) \int_0^1 \lambda \partial_j (x_m B_n(\lambda\vec{x})) d\lambda \\ &= \hat{e}_i \left(-\int_0^1 \lambda \partial_j (x_i B_j) d\lambda + \int_0^1 \lambda \partial_j (x_j B_i) d\lambda \right) \\ &= \hat{e}_i \int_0^1 d\lambda [-\lambda B_i - x_i \partial_j B_j + \lambda(3B_i + x_j \partial_j B_i)] \\ &= \hat{e}_i \int_0^1 d\lambda [2\lambda B_i + \lambda x_j \partial_j B_i] \end{aligned}$$

将第一项分部积分为:

$$\begin{aligned} 2\lambda B_i &= \frac{d(\lambda^2 B_i)}{d\lambda} - \lambda^2 \frac{dB_i}{d\lambda} \\ \frac{B_i(\lambda\vec{x})}{d\lambda} &= \frac{\partial B_i(\lambda\vec{x})}{\partial(\lambda x_j)} \frac{d(\lambda x_j)}{d\lambda} = \frac{x_j}{\lambda} \partial_j B_i \end{aligned}$$

Solution

故：

$$\begin{aligned}\nabla \times \vec{A} &= \hat{e}_i \int_0^1 d\lambda \left[\frac{d(\lambda^2 B_i)}{d\lambda} - \lambda^2 \frac{dB_i}{d\lambda} + \lambda x_j \partial_j B_i \right] \\ &= \lambda^2 \vec{B}(\lambda \vec{x}) \Big|_0^1 \\ &= \vec{B}(\vec{x})\end{aligned}$$