

Contents

- ✧ - Large-scale galaxy bias by Desjaques, Jeong, and Schmidt (2018), Appendix B
- ✧ - Donghui Jeong's thesis (2010), Chapter 2
- ✧ - Donghui Jeong's thesis (2010), Appendix E

Appendix B Cosmological perturbation theory

In this appendix, we review the PT approach to the description of the nonlinear gravitational evolution of dark matter density and velocity fields. On large scales, where the perturbative bias expansion is valid, the vorticity as well as baryonic pressure effects are small, and the nonlinear evolution of the cosmic matter density field can be well approximated by a self-gravitating, pressureless, longitudinal-velocity single fluid. We will turn to the corrections induced by neglecting the aforementioned effects in Appendix B.3.

Appendix B.1 Standard perturbation theory

In the pressureless-fluid approximation, the time evolution of the density $\delta(\mathbf{x}, \tau)$ and velocity fields $\mathbf{v}(\mathbf{x}, \tau)$ are governed by the continuity, Euler and Poisson equations:

$$\frac{\partial}{\partial \tau} \delta(\mathbf{x}, \tau) + \nabla \cdot \{[1 + \delta(\mathbf{x}, \tau)] \mathbf{v}(\mathbf{x}, \tau)\} = 0 \quad (\text{B.1})$$

$$\frac{\partial}{\partial \tau} \mathbf{v}(\mathbf{x}, \tau) + [\mathbf{v}(\mathbf{x}, \tau) \cdot \nabla] \mathbf{v}(\mathbf{x}, \tau) + \mathcal{H}(\tau) \mathbf{v}(\mathbf{x}, \tau) = -\nabla \Phi(\mathbf{x}, \tau) \quad (\text{B.2})$$

$$\nabla^2 \Phi(\mathbf{x}, \tau) = \frac{3}{2} \mathcal{H}^2 \Omega_m(\tau) \delta(\mathbf{x}, \tau). \quad (\text{B.3})$$

These equations are strictly valid for Λ CDM only; however, they are also highly accurate for canonical, non-clustering dark energy models where the sound horizon of the dark energy component is of order $\mathcal{H}^{-1}$ (see Sec. 8.3 for a brief discussion).

In Fourier space, we can eliminate the gravitational potential Φ and express the longitudinal velocity field in terms of the velocity divergence $\theta \equiv \nabla \cdot \mathbf{v}(\mathbf{x}, \tau)$ to obtain the following two equations:

$$\frac{\partial \delta(\mathbf{k}, \tau)}{\partial \tau} + \theta(\mathbf{k}, \tau) = - \int_{\mathbf{k}_1} \int_{\mathbf{k}_2} (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \alpha(\mathbf{k}_1, \mathbf{k}_2) \theta(\mathbf{k}_1, \tau) \delta(\mathbf{k}_2, \tau), \quad (\text{B.4})$$

$$\frac{\partial \theta(\mathbf{k}, \tau)}{\partial \tau} + \mathcal{H}(\tau) \theta(\mathbf{k}, \tau) + \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) \delta(\mathbf{k}, \tau) = - \int_{\mathbf{k}_1} \int_{\mathbf{k}_2} (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \beta(\mathbf{k}_1, \mathbf{k}_2) \theta(\mathbf{k}_1, \tau) \theta(\mathbf{k}_2, \tau). \quad (\text{B.5})$$

where $\mathbf{k}_{ij\dots} \equiv \mathbf{k}_i + \mathbf{k}_j + \dots$, and

$$\alpha(\mathbf{k}_1, \mathbf{k}_2) = \frac{\mathbf{k}_{12} \cdot \mathbf{k}_1}{k_1^2}, \quad \beta(\mathbf{k}_1, \mathbf{k}_2) = \frac{k_{12}^2 (\mathbf{k}_1 \cdot \mathbf{k}_2)}{2k_1^2 k_2^2}. \quad (\text{B.6})$$

In SPT, Eqs. (B.4)–(B.5) are solved perturbatively in terms of the linear density contrast $\delta^{(1)}(\mathbf{k}, \tau)$:

$$\delta(\mathbf{k}, \tau) = \sum_{n=1}^{\infty} \delta^{(n)}(\mathbf{k}, \tau), \quad \theta(\mathbf{k}, \tau) = \sum_{n=1}^{\infty} \theta^{(n)}(\mathbf{k}, \tau), \quad (\text{B.7})$$

where the superscript (n) indicates that a term involves n powers of the linear density contrast; we refer to such a term as being *n-th order in perturbation theory*.

On very large scales, $k \rightarrow 0$, $\delta^{(1)}$ is much less than 1, and the quadratic source term on the right-hand side of Eqs. (B.4)–(B.5) can be neglected. Then, all Fourier modes evolve independently, and we obtain a single second-order ordinary differential equation for the evolution of the linear density contrast, $\delta^{(1)}(\mathbf{k}, \tau)$:

$$\frac{\partial^2}{\partial \tau^2} \delta^{(1)}(\mathbf{k}, \tau) + \mathcal{H}(\tau) \frac{\partial}{\partial \tau} \delta^{(1)}(\mathbf{k}, \tau) - \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2(\tau) \delta^{(1)}(\mathbf{k}, \tau) = 0. \quad (\text{B.8})$$

We see that the time evolution equation is scale-independent; that is, all Fourier modes evolve at the same rate, independent of $\mathbf{k}$. In fact, this is the reason why time derivatives of the density field can be reordered to be successively higher order in perturbation theory in Sec. 2.5.2–2.5.3. We can then factor out the *linear*

growth factor $D(\tau)$ by writing $\delta^{(1)}(\mathbf{k}, \tau) = [D(\tau)/D(\tau_0)]\delta^{(1)}(\mathbf{k}, \tau_0)$, where τ_0 is a reference time, and $D(\tau)$ obeys

$$\frac{d^2}{d\tau^2}D(\tau) + \mathcal{H}\frac{d}{d\tau}D(\tau) - \frac{3}{2}\Omega_m(\tau)\mathcal{H}^2D(\tau) = 0, \quad (\text{B.9})$$

where initial conditions are setup so that $D(\tau)$ follows the growing mode, which corresponds to $D(\tau) \propto a(\tau)$ during matter domination. Unless otherwise indicated, all relations in this review are independent of the overall normalization of the growth factor. For reference, two popular choices for the latter are *i)* $D(\tau) = a(\tau)$ during matter domination (for $a \ll 1$), which is denoted as $D_{\text{md}}(\tau)$ in the text; and *ii)* $D(\tau_0) = 1$, where τ_0 corresponds to today's epoch.

The linear-theory velocity divergence field is then obtained from the linearized continuity equation [Eq. (B.4)],

$$\theta^{(1)}(\mathbf{k}, \tau) = -\frac{d \ln D(\tau)}{d\tau} \delta^{(1)}(\mathbf{k}, \tau) = -f\mathcal{H}(\tau)\delta^{(1)}(\mathbf{k}, \tau), \quad (\text{B.10})$$

where the logarithmic growth rate $f(\tau) \equiv d \ln D / d \ln a$. Note that the identity

$$\frac{d(\mathcal{H}f)}{d\tau} = \mathcal{H}^2 \left(\frac{3}{2}\Omega_m - f - f^2 \right) \quad (\text{B.11})$$

follows from the linear equations of motion.

The equations of motion Eqs. (B.4)–(B.5) motivate an ansatz of writing the n -th order solution in Eq. (B.7) as

$$\begin{aligned} \delta^{(n)}(\mathbf{k}, \tau) &= \int_{\mathbf{k}_1} \cdots \int_{\mathbf{k}_n} (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}_{12\dots n}) F_n(\mathbf{k}_1, \dots, \mathbf{k}_n, \tau) \delta^{(1)}(\mathbf{k}_1, \tau) \cdots \delta^{(1)}(\mathbf{k}_n, \tau) \\ \theta^{(n)}(\mathbf{k}, \tau) &= -\mathcal{H}(\tau)f(\tau) \int_{\mathbf{k}_1} \cdots \int_{\mathbf{k}_n} (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}_{12\dots n}) G_n(\mathbf{k}_1, \dots, \mathbf{k}_n, \tau) \delta^{(1)}(\mathbf{k}_1, \tau) \cdots \delta^{(1)}(\mathbf{k}_n, \tau), \end{aligned} \quad (\text{B.12})$$

with symmetrized density and velocity divergence kernels, respectively, F_n and G_n . From the linear solutions, $F_1 = G_1 = 1$ is obvious. Inserting the ansatz Eq. (B.12) into Eqs. (B.4)–(B.5) yields evolution equations for the kernels (see, e.g. [250]). In general, the kernels F_n and G_n depend on time, except when Ω_m/f^2 is constant in time. This is indeed the case for the EdS Universe where $\Omega_m = f = 1$ and the kernels become particularly simple. The usual practice in SPT is to calculate the time-independent kernels in the EdS Universe and apply the same kernels to other cosmologies. In this case, the linear growth function encodes the entire cosmology dependence of the nonlinear solutions in SPT. For Λ CDM and quintessence cosmologies, this approach provides a very accurate description of the full solution [858]. Specifically, the second-order kernels F_2 and G_2 in the EdS cosmology are given by

$$\begin{aligned} F_2(\mathbf{k}_1, \mathbf{k}_2) &= \frac{5}{7} + \frac{2}{7} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} + \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{2k_1 k_2} \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right), \\ G_2(\mathbf{k}_1, \mathbf{k}_2) &= \frac{3}{7} + \frac{4}{7} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} + \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{2k_1 k_2} \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right). \end{aligned} \quad (\text{B.13})$$

The second-order density and velocity field can also be written in real space as

$$\delta^{(2)}(\mathbf{x}, \tau) = \frac{17}{21}[\delta^{(1)}]^2 + \frac{2}{7}[K_{ij}^{(1)}]^2 - s_{(1)}^i \partial_i \delta^{(1)}, \quad (\text{B.14})$$

$$-\frac{1}{\mathcal{H}(\tau)f(\tau)}\theta^{(2)}(\mathbf{x}, \tau) = \frac{13}{21}[\delta^{(1)}]^2 + \frac{4}{7}[K_{ij}^{(1)}]^2 - s_{(1)}^i \partial_i \delta^{(1)}, \quad (\text{B.15})$$

where on the right-hand side of Eq. (B.14) all terms are evaluated at $\mathbf{x}, \tau$. Here,

$$\mathbf{s}_{(1)}(\mathbf{q}, \tau) = \mathbf{x}_{\text{fl}}^{(1)}(\tau) - \mathbf{q} = -\frac{\nabla}{\nabla^2} \delta^{(1)}(\mathbf{q}, \tau) \quad (\text{B.16})$$

is the first-order Lagrangian displacement, and

$$K_{ij}^{(1)} = \left[\frac{\partial_i \partial_j}{\nabla^2} - \frac{1}{3} \delta_{ij} \right] \delta^{(1)} \quad (\text{B.17})$$

is the first-order tidal field [Eq. (2.21)]. The third-order expressions in real space are given in Eq. (B.46) together with Eq. (B.48) below.

Finally, we give the result for the next-to-leading (NLO, or 1-loop) contribution to the matter and velocity power spectra. We will drop the time argument for clarity. It can be easily restored in the linear power spectra and perturbation theory kernels, if desired. The NLO matter power spectrum is given by $P_{mm}(k) = P_L(k) + P_{mm}^{\text{NLO}}(k)$, where

$$P_{mm}^{\text{NLO}}(k) = P_{mm}^{(22)}(k) + 2P_{mm}^{(13)}(k), \quad (\text{B.18})$$

$$P_{mm}^{(22)}(k) \equiv \langle \delta^{(2)}(\mathbf{k}) \delta^{(2)}(\mathbf{k}') \rangle' = 2 \int_{\mathbf{p}} [F_2(\mathbf{p}, \mathbf{k} - \mathbf{p})]^2 P_L(p) P_L(|\mathbf{k} - \mathbf{p}|) = \mathcal{I}^{[\delta^{(2)}, \delta^{(2)}]}(k)$$

$$P_{mm}^{(13)}(k) \equiv \langle \delta^{(1)}(\mathbf{k}) \delta^{(3)}(\mathbf{k}') \rangle' = 3P_L(k) \int_{\mathbf{p}} F_3(\mathbf{p}, -\mathbf{p}, \mathbf{k}) P_L(p), \quad (\text{B.19})$$

where $\mathcal{I}^{[O, O']}(k)$ is defined in Eq. (4.23), and the explicit expression for $P_{mm}^{(13)}(k)$ can be found in [859]. Similarly, the NLO matter-velocity cross-power spectrum and velocity-velocity power spectrum are given by $P_{m\theta}(k) = -f\mathcal{H}P_L(k) + P_{m\theta}^{\text{NLO}}(k)$ and $P_{\theta\theta}(k) = (f\mathcal{H})^2 P_L(k) + P_{\theta\theta}^{\text{NLO}}(k)$, where

$$P_{m\theta}^{\text{NLO}}(k) = P_{m\theta}^{(22)}(k) + 2P_{m\theta}^{(13)}(k), \quad (\text{B.20})$$

$$P_{m\theta}^{(22)}(k) \equiv \langle \delta^{(2)}(\mathbf{k}) \theta^{(2)}(\mathbf{k}') \rangle' = -2\mathcal{H}f \int_{\mathbf{p}} F_2(\mathbf{p}, \mathbf{k} - \mathbf{p}) G_2(\mathbf{p}, \mathbf{k} - \mathbf{p}) P_L(p) P_L(|\mathbf{k} - \mathbf{p}|)$$

$$P_{m\theta}^{(13)}(k) \equiv \frac{1}{2} \left[\langle \theta^{(1)}(\mathbf{k}) \delta^{(3)}(\mathbf{k}') \rangle' + \langle \delta^{(1)}(\mathbf{k}) \theta^{(3)}(\mathbf{k}') \rangle' \right] = -\frac{1}{2} \left[(\mathcal{H}f) P_{mm}^{(13)}(k) + (\mathcal{H}f)^{-1} P_{\theta\theta}^{(13)}(k) \right]$$

$$P_{\theta\theta}^{\text{NLO}}(k) = P_{\theta\theta}^{(22)}(k) + 2P_{\theta\theta}^{(13)}(k), \quad (\text{B.21})$$

$$P_{\theta\theta}^{(22)}(k) \equiv \langle \theta^{(2)}(\mathbf{k}) \theta^{(2)}(\mathbf{k}') \rangle' = 2(\mathcal{H}f)^2 \int_{\mathbf{p}} [G_2(\mathbf{p}, \mathbf{k} - \mathbf{p})]^2 P_L(p) P_L(|\mathbf{k} - \mathbf{p}|)$$

$$P_{\theta\theta}^{(13)}(k) \equiv \langle \theta^{(1)}(\mathbf{k}) \theta^{(3)}(\mathbf{k}') \rangle' = 3(\mathcal{H}f)^2 P_L(k) \int_{\mathbf{p}} G_3(\mathbf{p}, -\mathbf{p}, \mathbf{k}) P_L(p). \quad (\text{B.22})$$

The accuracy of the SPT predictions for the nonlinear matter power spectrum and bispectrum has been studied extensively using simulations [283, 860, 861, 287, 862, 248, 101, 127, 269, 612, 863]. Moreover, Ref. [101] showed that the accuracy improves significantly at higher redshifts ($z > 1$) where the nonlinear scale is smaller (see below).

Appendix B.2 Feynman rules of large-scale structure

The calculation of matter and galaxy statistics in perturbation theory can be performed efficiently by using a diagrammatic representation. The Feynman rules are as follows (see [344] for a detailed description):

1. An n -point correlation function is represented by a collection of diagrams with n outgoing external legs.
2. Interaction vertices have $m \geq 2$ ingoing lines $\mathbf{p}_1, \dots, \mathbf{p}_m$ coupling to a single outgoing line $\mathbf{p}$. Each such vertex is assigned a factor

$$m! F_m(\mathbf{p}_1, \dots, \mathbf{p}_m) (2\pi)^3 \delta_D(\mathbf{p} - \mathbf{p}_{1\dots m}). \quad (\text{B.23})$$

One assigns a positive (negative) sign to outgoing (ingoing) momenta. Each ingoing line has to be

vironment, they can in principle be calculated numerically by following a peak-background split argument similar to the one discussed in Sec. 3. Further, the effective stress tensor τ_{ij} can be expanded up to any order in terms of a basis of counter-terms that is constructed out of the same $\Pi_{ij}^{[n]}$ that are employed in the general Eulerian bias expansion in Sec. 2.5.3 [344].

Up to third order in perturbations, which is the highest order that is relevant for the expressions given in this review, there is only a single EFT contribution:

$$\delta_{\Lambda}^{(3)} \Big|_{\text{c.t.}}(\mathbf{k}, \tau) = -\alpha(\tau, \Lambda) D^3(\tau) \frac{k^2}{k_{\text{NL}}^2} \delta^{(1)}(\mathbf{k}, \tau) W_{\Lambda}(\mathbf{k}), \quad (\text{B.27})$$

where “c.t.” stands for counter-term, W_{Λ} denotes the smoothing kernel used to integrate out the small-scale modes, and k_{NL} is the nonlinear scale which we will define below. The free coefficient α cannot be predicted from perturbation theory and needs to be determined by fitting to simulations or observations. Essentially, it corresponds to an effective sound speed. We can now choose a vanishingly small smoothing scale $\Lambda^{-1} \rightarrow 0$, in which case we obtain another contribution to the NLO matter power spectrum [Eq. (B.18)],

$$P_{mm}^{\text{NLO}} \Big|_{\text{c.t.}}(k, \tau) = -2D^2(\tau)(2\pi)c_s^2 \frac{k^2}{k_{\text{NL}}^2} P_{\text{L}}(k, \tau), \quad (\text{B.28})$$

where we have traded $\lim_{\Lambda^{-1} \rightarrow 0} \alpha(\Lambda)$ for c_s^2 . This results in the contribution given in Eq. (4.22), where we have absorbed the growth factor $D^2(\tau)$ into $c_{s,\text{eff}}^2$. Furthermore, by allowing c_s to be fit to simulations, this correction term absorbs the dependence of the SPT contribution $P_{mm}^{(13)}$ in Eq. (B.18) on fully nonlinear small-scale modes that cannot be described by perturbation theory. Empirically, one finds values for c_s of order 1 (e.g., [85]). Note that the leading stochastic EFT contribution to the matter power spectrum scales as $k^4 P_{\varepsilon_m}^{\{4\}}(\tau)$ [346], where $\varepsilon_m \propto p_{\text{eff}}$ and $P_{\varepsilon_m}^{\{4\}}(\tau)$ is its power spectrum amplitude in the low- k limit which only depends on time. The scaling with k^4 is required by mass and momentum conservation [71, 344]. In terms of the perturbative counting of terms, this is a higher-order contribution; however, ε_m is relevant in the NLO halo-matter cross-power spectrum (Sec. 4.1.4).

We now turn to the general expectation for how large higher-order nonlinear terms to the n -point functions of matter and biased tracers are at a given scale k . The most important scale is the *nonlinear scale* which we here define as the scale where the dimensionless matter power spectrum $k^3 P_{\text{L}}(k)/(\pi^2)$ becomes unity. Note that various different choices are possible here, since this is only a rough estimate for the scale where higher-order corrections become of order one. For our reference cosmology, this corresponds to $k_{\text{NL}}(z=0) \simeq 0.25 h \text{ Mpc}^{-1}$. At and below this scale, $P_{\text{L}}(k)$ can be approximated by a power law $P_{\text{L}}(k) \propto k^n$ with $n = -1.9 \dots -1.7$ depending on the precise fit range chosen. As discussed in Sec. 4.1.4, higher-order SPT contributions to the n -point functions roughly scale as $(k/k_{\text{NL}})^{n+3} \sim (k/k_{\text{NL}})^{1.1 \dots 1.3}$, while the leading EFT term scales as $(k/k_{\text{NL}})^2$, for $c_s = \mathcal{O}(1)$.

Appendix B.4 IR resummation

The perturbative expansion and counting of terms discussed so far in this Appendix (as well as in Sec. 4.1.4) relies on the assumption of a smooth, slowly varying linear matter power spectrum. In reality, the plasma oscillations in the baryon-photon fluid prior to recombination imprint a small-amplitude oscillatory pattern in the linear matter power spectrum, the well-known BAO feature first pointed out by [41], with oscillation period $k_{\text{BAO}} = 2\pi/r_s \approx 0.06 h \text{ Mpc}^{-1}$, where $r_s \approx 106 h^{-1} \text{ Mpc}$ is the sound horizon at recombination. These oscillations are damped towards higher k by Silk damping.

Now consider a perturbation with wavenumber $p \sim k_{\text{BAO}}$. This mode induces a displacement $\mathbf{s}$ which moves small-scale perturbations with $k \gg p$. Perturbatively, this effect is captured at second order by the contribution $\mathbf{s}^i \partial_i \delta$. However, the effect of higher-order displacement terms on the oscillatory (“wiggles”) part of the power spectrum is not suppressed, since the displacement itself is modulated over scales of order r_s . Fortunately, since this is a pure displacement effect, i.e., a coordinate transformation, these terms can be resummed [269, 270, 271, 272]. This resummation relies on the fact that (i) the relevant modes sourcing the displacement are in the perturbative regime ($p \lesssim 0.2 h \text{ Mpc}^{-1}$), and (ii) that the shape of the oscillatory

feature is well known. Crucially, only the long-wavelength displacement effects on this oscillatory feature are resummed, as they are the ones which are enhanced.

First, we split the linear power spectrum into smooth (s) and oscillatory parts (w) via

$$P_L(k) = P_L^s(k) + P_L^w(k), \quad (\text{B.29})$$

where

$$P_L^w(k) = \sin\left(\frac{k}{k_{\text{BAO}}}\right) f_{\text{env}}(k) \quad (\text{B.30})$$

is a strictly oscillating function whose amplitude is controlled by the smooth envelope $f_{\text{env}}(k)$, while $P_L^s(k)$ is defined as the remainder. Then, the IR resummation acting on the oscillatory part yields [272]²⁷ (see [270, 271, 864] for similar results)

$$\begin{aligned} P_{mm}^{\text{IR-resum.}}(k) &= P_L^s(k) + \exp[-\Sigma^2(\epsilon k)k^2] (1 + \Sigma^2(\epsilon k)k^2) P_L^w(k) \\ &+ P_{mm}^{\text{NLO}}[P_L(k') \rightarrow P_L^s(k') + \exp(-\Sigma^2(\epsilon k')k'^2) P_L^w(k')] (k). \end{aligned} \quad (\text{B.31})$$

Here, $P_{mm}^{\text{NLO}}[\dots](k)$ stands for the NLO matter power spectrum Eq. (B.18) evaluated using the IR-resummed version of the linear power spectrum. Finally,

$$\Sigma^2(\Lambda) = \frac{1}{6\pi^2} \int_0^\Lambda dp P_L(p) [1 - j_0(pr_s) + 2j_2(pr_s)] \quad (\text{B.32})$$

is essentially the covariance $\langle s^i s^j \rangle$ of the linear Lagrangian displacement, after angle-averaging and smoothing with a sharp- k filter at the scale Λ . $\epsilon \ll 1$ is a parameter controlling down to what scale the IR resummation should occur; Ref. [271] choose $\epsilon = 1/2$. The term $+\Sigma^2 k^2$ in parentheses in the first line of Eq. (B.31) prevents the double counting of terms that appear in the NLO expression on the second line. As shown in [270, 271, 864, 272], the IR-resummed matter power spectrum and its Fourier transform, the correlation function, significantly improve the match in the shape of these correlations with full N-body simulations on the scales around the BAO feature ($r \sim r_s$ in real space; $0.05 h \text{ Mpc}^{-1} \lesssim k \lesssim 0.3 h \text{ Mpc}^{-1}$ in Fourier space).

At this, leading order in the wavelength $1/p$ of the soft modes that lead to displacements, the displacements are the same for both matter and LSS tracers, due to the equivalence principle (Sec. 2.7; see also [865]). This means that we can immediately make use of the IR resummation in the prediction for halo and galaxy clustering in real space, by generalizing Eq. (B.31) to:

$$\begin{aligned} P_{hm}^{\text{IR-resum.}}(k) &= b_1 P_L^s(k) + b_1 \exp[-\Sigma^2(\epsilon k)k^2] (1 + \Sigma^2(\epsilon k)k^2) P_L^w(k) \\ &+ P_{hm}^{\text{NLO}}[P_L(k') \rightarrow P_L^s(k') + \exp(-\Sigma^2(\epsilon k')k'^2) P_L^w(k')] (k), \end{aligned} \quad (\text{B.33})$$

and

$$\begin{aligned} P_{hh}^{\text{IR-resum.}}(k) &= b_1^2 P_L^s(k) + b_1^2 \exp[-\Sigma^2(\epsilon k)k^2] (1 + \Sigma^2(\epsilon k)k^2) P_L^w(k) + P_\epsilon^{\{0\}} \\ &+ P_{hh}^{\text{NLO}}[P_L(k') \rightarrow P_L^s(k') + \exp(-\Sigma^2(\epsilon k')k'^2) P_L^w(k')] (k), \end{aligned} \quad (\text{B.34})$$

where the second line in each equation is obtained by replacing the linear power spectrum with its IR-resummed version in Eq. (4.22). Again, this is because the IR resummation only deals with displacement terms which, at leading order in derivatives, are only modified by biasing through the trivial factor b_1 . This holds analogously for the displacement from the galaxy rest frame into redshift space (Sec. 9.3.2), whose IR contributions can similarly be resummed. Refs. [89, 180, 866, 867, 865] provide resummed results for two-point functions in redshift space.

²⁷Eq. (7.4) there; we have dropped the 2-loop-order displacement terms $\propto D^6$ in that expression, since they are numerically small. Note also the different Fourier convention adopted in [272].

Appendix B.5 Convective SPT approach and conserved evolution at third order

In this section, we provide some details on the calculation of the evolution of a conserved biased tracer at third order in perturbation theory that is briefly presented in Sec. 2.4. The “convective SPT” system, consisting of the continuity and Euler equation Eqs. (B.1)–(B.2) for matter and the continuity equation Eq. (2.17) for the biased tracer, can be written in compact form as

$$\frac{D}{D\tau} \Psi = -\sigma \cdot \Psi + S \quad (\text{B.35})$$

where

$$\Psi(\mathbf{x}, \tau) = \begin{pmatrix} \delta_g(\mathbf{x}, \tau) \\ \delta(\mathbf{x}, \tau) \\ \theta(\mathbf{x}, \tau) \end{pmatrix}; \quad \sigma(\tau) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & \frac{3}{2}\Omega_m \mathcal{H}^2 & \mathcal{H} \end{pmatrix}; \quad S(\mathbf{x}, \tau) = \begin{pmatrix} -\delta_g \theta \\ -\delta \theta \\ -(\partial_i v_j)^2 \end{pmatrix}. \quad (\text{B.36})$$

The fact that σ is degenerate already shows that the three equations are not really coupled, but rather the equation for δ_g can be integrated separately as done in Sec. 2.3. Eq. (B.35) allows for a convenient, compact derivation of both δ and δ_g . In the following, we will solve this system up to third order.

Source term

The goal is to integrate Eq. (B.35) along the fluid trajectory. However, the source term S involves some subtleties, in particular the third component $S_3 = (\partial_i v_j)^2$, since the derivative is with respect to Eulerian coordinate $\mathbf{x}$. The velocity shear $\partial_i v_j$ is nonlocally related to the degrees of freedom δ, θ themselves [since $\mathbf{v} = (\nabla/\nabla^2)\theta$]. We can use a trick, namely the fact that the displacement is the integral of the peculiar velocity along the fluid trajectory $\mathbf{x}_\text{fl}(\mathbf{q}, \tau)$, and therefore

$$\mathbf{v}(\mathbf{x}_\text{fl}[\mathbf{q}, \tau], \tau) = \frac{D}{D\tau} \mathbf{s}(\mathbf{q}, \tau). \quad (\text{B.37})$$

In the remainder of this section, we will abbreviate convective derivatives $D/D\tau$ as primes. Further, we need to transform the derivative from $\mathbf{x}$ to the fluid flow or Lagrangian coordinate $\mathbf{q}$ via

$$\partial_x^i = \left([1 + \mathbf{M}]^{-1}\right)_j^i \partial_q^j, \quad (\text{B.38})$$

where $M^{ij} = \partial_q^i s^j$. At second order, we then obtain

$$\partial_x^i v^j = [\partial_q^i + (\partial_q^i s_k) \partial_q^k] s'^j + \mathcal{O}([\delta^{(1)}]^3). \quad (\text{B.39})$$

Here, the left-hand side is defined at the Eulerian coordinate $(\mathbf{x}, \tau)$ while the right-hand side quantities are defined at the corresponding Lagrangian coordinate $\mathbf{q}$. With these expressions, we can construct source terms that are expressed purely in terms of convective time and Lagrangian spatial derivatives. Specifically, the third component of the source term, S_3 , is at first and second order given by (App. B in [131])

$$\begin{aligned} (S^{(1)})_3 &= -[a'(\tau)]^2 \left[(K^{(1)})^2 + \frac{1}{3}(\delta^{(1)})^2 \right]_{\tau_0} \\ (S^{(2)})_3 &= 2a[a'(\tau)]^2 \left[-\frac{2}{3}\delta^{(1)}\sigma^{(2)} - 2K_{ij}^{(1)}\mathcal{D}^{ij}\sigma^{(2)} + \frac{1}{9}(\delta^{(1)})^3 + \delta^{(1)}(K^{(1)})^2 + (K^{(1)})^3 \right]_{\tau_0}, \end{aligned} \quad (\text{B.40})$$

where

$$\sigma^{(2)} \equiv \partial_k s^{(2)k} = \frac{1}{2} \left[-\frac{2}{7}(\delta^{[1]})^2 + \frac{3}{7}(K^{[1]})^2 \right] \quad (\text{B.41})$$

is the divergence of the second-order Lagrangian displacement.

Initial conditions

In order to integrate Eq. (B.35), we need to provide an expression for the initial conditions at some initial time τ_* . The quantities δ^* , θ^* are obtained by integrating only the matter part $[\Psi]_{i=2,3}$ from $\tau = 0$ to $\tau = \tau_*$. Throughout, we only consider the fastest-growing-mode solutions for the density and velocity field. That is, the final time τ is assumed to be late enough so that all the slowly growing modes are subdominant [503, 505].

Under the assumptions explained in Sec. 2.4, the galaxy density δ_g^* at the “formation time” is given by Eq. (2.44), which we repeat here for convenience:

$$\begin{aligned} \delta_g^* \equiv \delta_g(\mathbf{x}_\text{fl}(\tau_*), \tau_*) &= \sum_{n=1}^3 \frac{b_n^*}{n!} [\delta^*]^n + b_{K^2}^* \text{tr} [(K_{ij}^*)^2] + b_{K^3}^* \text{tr} [(K_{ij}^*)^3] + b_{\delta K^2} \delta^* \text{tr} [(K_{ij}^*)^2] \\ &+ \varepsilon^* + \varepsilon_\delta^* \delta^* + \varepsilon_{\delta^2}^* [\delta^*]^2 + \varepsilon_{K^2}^* \text{tr} [(K_{ij}^*)^2] , \end{aligned} \quad (\text{B.42})$$

where here and throughout a superscript $*$ indicates that a quantity is evaluated at $\mathbf{x}_* \equiv \mathbf{x}_\text{fl}(\tau_*)$ and τ_* . The stochastic fields ε^* , ε_X^* ($X = \delta, \delta^2, K^2$) are assumed to be first-order random variables.

Solution

We define the matrix $\mathbf{A}(\tau, \tau_i)$ as the solution to the following matrix ODE with boundary condition

$$\partial_\tau \mathbf{A}(\tau, \tau_i) + \boldsymbol{\sigma}(\tau) \mathbf{A}(\tau, \tau_i) = 0; \quad \mathbf{A}(\tau, \tau) = \mathbf{1}. \quad (\text{B.43})$$

In order to obtain closed analytical expressions, we restrict to an EdS Universe. In this case, $\mathbf{A}$ assumes a simple form constructed out of the growing and decaying modes which scale as $a(\tau) \propto \tau^2$ and $H(\tau) \propto \tau^{-3}$, respectively (App. B in [131]). In the final result, we then replace scale factors $a(\tau)$ with growth factors $D(\tau)$, as described after Eq. (B.12).

Given the initial condition for Eq. (B.35) from above, $\Psi(\mathbf{x}_*, \tau_*) = (\delta_g^*, \delta^*, \theta^*)$ with Eq. (B.42), the particular solution to Eq. (B.35) is then given by

$$\Psi(\mathbf{x}, \tau) = \mathbf{A}(\tau, \tau_*) \begin{pmatrix} \delta_g^* \\ \delta^* \\ \theta^* \end{pmatrix} + \int_{\tau_*}^{\tau} d\tau' \mathbf{A}(\tau, \tau') \mathbf{S}(\mathbf{x}_\text{fl}(\tau'), \tau'). \quad (\text{B.44})$$

The interpretation of Eq. (B.44) is clear: the density and velocity of the fluid and galaxy at position $(\mathbf{x}, \tau)$ is given by an integral over the fluid trajectory of the source term multiplied by the Green’s function $\mathbf{A}(\tau, \tau')$. Eq. (B.44) leads to the SPT result for matter, in particular for the terms that are invariant under time-dependent coordinate shifts $\mathbf{x} \rightarrow \mathbf{x} + \boldsymbol{\xi}(\tau)$. The non-invariant terms, which are not captured by Eq. (B.44), are precisely the displacement terms relating the solutions in Lagrangian and Eulerian coordinates. Finally, expanding in these displacements to the same perturbative order then yields exactly the SPT result. Thus, the convective SPT calculation keeps track of the displacement terms exactly (i.e., it resums them), similar to Lagrangian perturbation theory.

At second order, we then recover our previous result, Eq. (2.33) as well as Eqs. (B.14)–(B.15). Specifically, the solution reads

$$\Psi^{(2)}(\mathbf{x}, \tau) = \begin{pmatrix} b_1^E(\tau) \delta^{(2)} + b_2^E(\tau) (\delta^{(1)})^2 / 2 + b_{K^2}^E(\tau) (K_{ij}^{(1)})^2 + \varepsilon_\delta^E(\tau) \delta^{(1)} \\ D^2(\tau) \left[\frac{17}{21} (\delta^{(1)})^2 + \frac{2}{7} (K_{ij}^{(1)})^2 \right] \\ -D(\tau) \dot{D}(\tau) \left[\frac{13}{21} (\delta^{(1)})^2 + \frac{4}{7} (K_{ij}^{(1)})^2 \right] \end{pmatrix}_q. \quad (\text{B.45})$$

²⁸Note that $\theta^{(3)}$ in Eq. (B.23) of [131] lacks an overall minus sign.

At third order, we obtain for the matter fields [131]²⁸

$$\begin{aligned}\delta^{(3)} &= \frac{341}{567}\delta^3 + \frac{11}{21}K^2\delta + \frac{2}{9}K^3 + \frac{1}{6}O_{\text{td}} \\ \theta^{(3)} &= -f\mathcal{H}\left[\frac{71}{189}\delta^3 + \frac{5}{7}K^2\delta + \frac{2}{3}K^3 + \frac{1}{2}O_{\text{td}}\right],\end{aligned}\quad (\text{B.46})$$

where K^n stands for $\text{tr}[K_{ij}^n]$, and all quantities on the right-hand side are evaluated at linear order and at τ . O_{td} is defined in Eq. (2.50). Note that, although all quantities appearing in Eq. (B.46) are defined using derivatives with respect to $\mathbf{x}$, the distinction between Eulerian and Lagrangian derivatives in these terms only matters at higher order. Notice further that the coefficient of the δ^3 term in $\delta^{(3)}$ is exactly the third-order coefficient of the perturbative expansion of spherical collapse in an EdS Universe [Eq. (2.14)], as expected: for a spherically symmetric perturbation, $K_{ij} = 0$, and $\delta^{(3)}$ has to reduce to the cubic term in the perturbative expansion of the spherical collapse solution. To the authors' knowledge, despite its fairly simple form, Ref. [131] was the first to give the complete expression for $\delta^{(3)}$, $\theta^{(3)}$ in real space. Interestingly, the expression for the nonlinear density derived by Ref. [136] from ellipsoidal collapse [444, 435] does not contain a term of the form $K^{ij}\mathcal{D}_{ij}\sigma^{(2)}$, while it matches the PT result at second order. The same is found in the local tidal approximation (LTA) of [445], as well as the fully relativistic local approximation derived in [150]. The underlying reason is that the evolution equations of an isolated ellipsoidal perturbation are only approximate, since they neglect the dynamical interaction with the large-scale environment. When allowing for general configurations, the nonlinear evolution of tidal fields in cosmology is nonlocal [868, 148, 149].

Finally, in order to obtain the density at a fixed order in standard Eulerian perturbation theory, we need to displace Ψ from a fixed Lagrangian position to the Eulerian position, by expanding in the argument. Let us define the Eulerian solution Ψ^E through

$$\Psi^E(\mathbf{x}) = \Psi(\mathbf{x} - \mathbf{s}[\mathbf{q}, \tau]). \quad (\text{B.47})$$

Ψ on the right-hand side contains all invariant terms, which we have obtained above. We now perform a Taylor expansion in $\mathbf{s}$ as well as a perturbative expansion of $\mathbf{s}$, noting that $\mathbf{s}$ is itself a function of the Lagrangian rather than Eulerian position. We obtain at third order

$$\Psi^{E,(3)}(\mathbf{x}, \tau) = \Psi^{(3)} - s_{(1)}^i \partial_i \Psi^{(2)} - \left[s_{(2)}^i - s_{(1)}^j (\partial_j s_{(1)}^i) \right] \partial_i \Psi^{(1)} + \frac{1}{2} s_{(1)}^i s_{(1)}^j \partial_i \partial_j \Psi^{(1)}, \quad (\text{B.48})$$

where on the right-hand side all quantities are evaluated at $\mathbf{x}$, τ , and $\Psi^{(n)} = \{\delta_g^{(n)}, \delta^{(n)}, \theta^{(n)}\}$, are given in Eq. (B.45) for $n = 2$, and Eq. (2.49), Eq. (B.46) for $n = 3$, respectively. As expected, the displacement from Lagrangian to Eulerian position, being merely a coordinate shift, does not affect the bias relation at any order.

Appendix B.6 Conserved evolution and bias expansion beyond the EdS background

We now allow for a more general expansion history, in particular Λ CDM or quintessence (neglecting perturbations in the dark energy component), and derive how this changes the evolution equations of a conserved tracer in the convective SPT approach. For this, we transform the time coordinate in the equations of motion Eq. (2.43) to $\ln D$, where D is the linear growth factor [Eq. (B.9)]. Since $d \ln D / d\tau = \mathcal{H} d \ln D / d \ln a = \mathcal{H} f$ by definition of the growth rate f , we introduce

$$\tilde{\mathbf{v}} \equiv (\mathcal{H}f)^{-1} \mathbf{v} \quad \text{and} \quad \frac{D}{D \ln D} \equiv \frac{\partial}{\partial \ln D} + \tilde{v}^i \partial_i. \quad (\text{B.49})$$

Note that $\tilde{\theta} = \partial_i \tilde{v}^i = -\delta$ at linear order. Using Eq. (B.11), we then obtain

$$\frac{D}{D \ln D} \tilde{\Psi} = -\tilde{\sigma} \cdot \tilde{\Psi} + \tilde{S} \quad (\text{B.50})$$

$$\tilde{\Psi}(\mathbf{x}, \tau) = \begin{pmatrix} \delta_g(\mathbf{x}, \tau) \\ \delta(\mathbf{x}, \tau) \\ \tilde{\theta}(\mathbf{x}, \tau) \end{pmatrix}; \quad \tilde{\sigma}(\tau) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & \frac{3}{2} \frac{\Omega_m}{f^2} & \frac{3}{2} \frac{\Omega_m}{f^2} - 1 \end{pmatrix}; \quad \tilde{S}(\mathbf{x}, \tau) = \begin{pmatrix} -\delta_g \tilde{\theta} \\ -\delta \tilde{\theta} \\ -(\partial_i \tilde{v}_j)^2 \end{pmatrix}. \quad (\text{B.51})$$

We see that the equations maintain the same structure, and that the departure from EdS of the equations of motion is completely quantified by $\Omega_m(a)/f^2(a) - 1$. In particular, the time dependence of the second-order matter density field is, instead of $[D(\tau)]^2$, given by a second-order growth factor $D_2(\tau)$, obtained by integrating $[D(\tau)]^2$ against the Green's function corresponding to Eq. (B.51). Only starting at third order do we obtain two different time dependences in the contributions to δ at the same perturbative order; specifically, $D_{3,1}(\tau)$ and $D_{3,2}(\tau)$, obtained respectively by integrating $[D(\tau)]^3$ and $D_2(\tau)D(\tau)$ against the Green's function.

We now go back to the construction of the set of general local bias operators in Sec. 2.5.2–2.5.3. The first instance of a new operator being induced by the different time evolution in quintessence cosmology is at third order, where $\Pi_{ij}^{[3]}$ defined via Eq. (2.63) in Sec. 2.5.3 separates into two different operators $\Pi_{ij}^{[3,1]}$, $\Pi_{ij}^{[3,2]}$. Now, the reason for why we do not need to include convective time derivatives of δ itself, or equivalently $\text{tr}[\Pi^{[n]}]$, in the perturbative bias expansion also holds in the more general case considered here: by including time derivatives of all other operators constructed out of $\partial_i \partial_j \Phi$, we already obtain the operators appearing in the time derivatives of δ , or equivalently in the perturbative contributions $\delta^{(2)}$, $\delta^{(3)}$, $\dots$. Thus, we do not need to include $\text{tr}[\Pi^{[3,i]}]$, and the first new operator in the Eulerian basis Eq. (2.64) of the bias expansion appears at fourth order, where

$$\text{tr}[\Pi^{[1]}\Pi^{[3]}] \longrightarrow \text{tr}[\Pi^{[1]}\Pi^{[3,1]}], \text{tr}[\Pi^{[1]}\Pi^{[3,2]}]. \quad (\text{B.52})$$

Conversely, in the EdS limit these two operators have to combine into $\text{tr}[\Pi^{[1]}\Pi^{[3]}]$. This means that the bias coefficients of the two different fourth-order operators will be approximately equal, with a fractional difference scaling as $D_{3,1}(\tau)/D_{3,2}(\tau) - 1 \ll 1$. Thus, the bases of bias operators described in Sec. 2.5.2–2.5.3 can be straightforwardly extended beyond EdS to quintessence cosmologies. Moreover, all explicit results for galaxy and halo statistics derived in Sec. 4, which only rely on perturbative results up to third order, are independent of the EdS assumption. While the time evolution of bias parameters for conserved tracers [Eq. (2.52)] is modified from the EdS case, this is not of practical relevance if all the bias parameters are determined from the data at a fixed time.

Chapter 2

Eulerian cosmological Perturbation Theory

Cosmological Perturbation Theory (Bernardeau et al., 2002, and references therein) provides the unique theoretical framework of studying the evolution of the density and velocity fields of matter fluctuation in the Universe. While the non-linear gravitational instability breaks down the validity of linear Perturbation Theory on smaller scales ($k \gtrsim 0.1 [h/\text{Mpc}]$ at present), we expect to model the non-linear evolution of cosmic matter field by using higher order Perturbation Theory. Yet, there is **a fundamental limitation of Perturbation Theory: it improves upon the linear theory only in the very small region when non-linearity is too strong (this happens around $z \sim 0$), and breaks down on the scales where non-perturbative effects such as shell-crossing and violent relaxation take place. Therefore, we define *quasi-nonlinear regime* where higher-order Perturbation Theory correctly models the non-linear evolution of cosmic matter field.**

Quasi-nonlinear regime in standard Perturbation Theory satisfies following three conditions.

- [1] Quasi-nonlinear regime is small compare to the Hubble length so that evolution of cosmic matter field is governed by Newtonian fluid equations.
- [2] Quasi-nonlinear regime is large enough to neglect baryonic pressure so that we can treat dark matter and baryon as a single component of pressureless matter.
- [3] In quasi-nonlinear regime, vorticity developed by non-linear gravitational interaction is negligibly small.

With these three conditions, we can approximate cosmic matter field as a pressureless, single component Newtonian fluid which is completely described by its density contrast and velocity gradient. In Section 2.1, we shall present the Perturbation Theory calculation of the non-linear evolution of cosmic matter field based on these conditions.

Extended studies of standard Perturbation Theory by relaxing one of these conditions are also available in literature. Noh & Hwang (2008) have studied the single fluid

equation in the full General Relativistic context and show that, if one use the proper gauge (temporal comoving gauge, to be specific), the General Relativistic perturbation equations *exactly* coincide with their Newtonian counterparts up to 2nd order; thus, **the General Relativistic correction appears from third order in density perturbation**. In Noh et al. (2009), we showed that the 3rd order General Relativistic correction term is sub-dominant on sub-horizon scales, so that the Newtonian PT approach is valid on quasi-nonlinear regime. It is because **the purely General Relativistic effect comes through the gravitational potential, and the gravitational potential is much smaller than the density field on sub-horizon scales**. At the same time, this correction term increases on large scales comparable to the Hubble radius, because gravitational potential sharply increases as $P_\phi \propto k^{n_s-4}$; thus, it eventually exceeds the linear power spectrum near horizon scale. Shoji & Komatsu (2009) have included a pressureful component to the analysis and have found a perturbative solution of double-fluid equations up to 3rd order. Finally, Pueblas & Scoccimarro (2009) measures the vorticity power spectrum from N-body simulations, and show that vorticity effect on density power spectrum is indeed negligible in the quasi-nonlinear regime.

These studies have indicated that non-linear effects coming from violating three conditions are not significant on scales which are most relevant for upcoming high redshift galaxy surveys. One notable exception is when including massive neutrinos. Massive neutrinos suppress the linear power spectrum below the neutrino free streaming scale (Takada et al., 2006), and change nonlinear matter power spectrum, correspondingly. Although the non-linear effect to the matter power spectrum is marginal due to the small energy fraction of neutrino, $f_\nu \equiv \Omega_\nu/\Omega_m$, this effect has to be included in order to measure neutrino mass from galaxy surveys (Shoji & Komatsu, 2009; Saito et al., 2009).

Once we model the non-linear evolution of density field and velocity field of cosmic matter fluctuation, we can calculate the galaxy power spectrum we would observe from galaxy surveys. Here, we have to model two more non-linearities: nonlinear redshift space distortion and nonlinear bias. In order to understand those non-linear effects separately, we first present the non-linear galaxy power spectrum in real space in Section 2.5, then present the non-linear redshift space matter power spectrum in Section 2.6. We combine all the non-linearities and present the non-linear galaxy power spectrum in redshift space in Section 2.7. For each section, we also analyze the effect of primordial non-Gaussianity on the power spectrum of large scale structure in the Perturbation Theory framework.

While we focus on the Eulerian Perturbation Theory in this chapter, Lagrangian Perturbation Theory provide yet another intuition on the non-linear growth of the structure.

In particular, Lagrangian perturbation theory (or its linear version which is also known as *Zel'dovich approximation*) is widely used to generate the initial condition for cosmological N-body simulations. We review the Lagrangian Perturbation Theory in Appendix E.

Although the material in this chapter is self-contained, we by no means aim for the complete review. For more detailed review on Perturbation Theory, we refer readers to Bernardeau et al. (2002).

2.1 Eulerian Perturbation Theory solution

We review calculation of non-linear Eulerian Perturbation Theory following the pioneering work in the literature (Vishniac, 1983; Fry, 1984; Goroff et al., 1986; Suto & Sasaki, 1991; Makino et al., 1992; Jain & Bertschinger, 1994; Scoccimarro & Frieman, 1996). We treat dark matter and baryons as pressureless dust particles, as we are interested in the scales much larger than the Jeans length. We also assume that peculiar velocity is much smaller than the speed of light, which is always an excellent approximation, and that the fluctuations we are interested in are deep inside the horizon; thus, we treat the system as Newtonian. Then, the evolution of the matter fluctuation, $\delta(\mathbf{x}, \tau) \equiv \rho(\mathbf{x}, \tau)/\bar{\rho}(\tau) - 1$, follows Newtonian fluid equations in expanding universe:

$$\dot{\delta} + \nabla \cdot [(1 + \delta)\mathbf{v}] = 0, \quad (2.1)$$

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\mathcal{H}\mathbf{v} - \nabla\phi, \quad (2.2)$$

$$\nabla^2\phi = 4\pi G a^2 \bar{\rho} \delta, \quad (2.3)$$

where the dots denote $\partial/\partial\tau$ (τ is the conformal time), ∇ denotes $\partial/\partial\mathbf{x}$ ($\mathbf{x}$ is the comoving coordinate), $\mathbf{v} = d\mathbf{x}/d\tau$ is the peculiar velocity field, and ϕ is the peculiar gravitational potential field from density fluctuations, and $\mathcal{H} \equiv d\ln a/d\tau = aH$. As we ignore the vorticity, $\mathbf{v}$ is curl-free, which motivates our using $\theta \equiv \nabla \cdot \mathbf{v}$, the velocity divergence field.

In Fourier space, the Newtonian fluid equations become two coupled integro-differential equations for $\delta_{\mathbf{k}}(\tau)$ and $\theta_{\mathbf{k}}(\tau)$. Using equation (2.3) and the Friedmann equation, we write

the continuity equation [Eq. (2.1)] and the Euler equation [Eq. (2.2)] in Fourier space

$$\begin{aligned} & \frac{\partial \delta_{\mathbf{k}}}{\partial \tau}(\tau) + \theta_{\mathbf{k}}(\tau) \\ &= - \int \frac{d^3 k_1}{(2\pi)^3} \int d^3 k_2 \delta^D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \frac{\mathbf{k} \cdot \mathbf{k}_1}{k_1^2} \theta_{\mathbf{k}_1}(\tau), \delta_{\mathbf{k}_2}(\tau), \end{aligned} \quad (2.4)$$

$$\begin{aligned} & \frac{\partial \theta_{\mathbf{k}}}{\partial \tau}(\tau) + \mathcal{H}(\tau) \theta_{\mathbf{k}}(\tau) + \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) \delta_{\mathbf{k}}(\tau) \\ &= - \int \frac{d^3 k_1}{(2\pi)^3} \int d^3 k_2 \delta^D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \frac{k^2(\mathbf{k}_1 \cdot \mathbf{k}_2)}{2k_1^2 k_2^2} \theta_{\mathbf{k}_1}(\tau) \theta_{\mathbf{k}_2}(\tau), \end{aligned} \quad (2.5)$$

respectively. Note that left hand side of equations above are linear in perturbation variables, and non-linear evolution is described by the right hand side as coupling between different Fourier modes.

2.1.1 Linear solution for density field and velocity field

When density and velocity fluctuations are small, we can neglect the mode coupling terms in the right hand side of equation (2.4) and equation (2.5). Then, the continuity and the Euler equation are linearized as

$$\frac{\partial \delta_1(\mathbf{k}, \tau)}{\partial \tau} + \theta_1(\mathbf{k}, \tau) = 0, \quad (2.6)$$

$$\frac{\partial \theta_1(\mathbf{k}, \tau)}{\partial \tau} + \mathcal{H}(\tau) \theta_1(\mathbf{k}, \tau) + \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) \delta_1(\mathbf{k}, \tau) = 0. \quad (2.7)$$

Combining these two equations, we have a second order differential equation for $\delta_1(\mathbf{k}, \tau)$ as

$$\frac{\partial^2 \delta_1(\mathbf{k}, \tau)}{\partial \tau^2} + \mathcal{H}(\tau) \frac{\partial \delta_1(\mathbf{k}, \tau)}{\partial \tau} + \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) \delta_1(\mathbf{k}, \tau) = 0, \quad (2.8)$$

whose solution is given by

$$\delta_1(\mathbf{k}, a) = C_+(\mathbf{k}) H(a) \int_0^a \frac{da'}{a'^3 H(a')^3} + C_-(\mathbf{k}) H(a). \quad (2.9)$$

Here, the first term is a growing mode and the second term is a decaying mode.

Let us only consider a growing mode. There are two conventions in the literature about normalizing a growing mode. One normalization convention is requiring that a growing mode is equal to the scale factor in the matter dominated epoch: $D_+(a)|_{\text{EdS}} = a$. Here, EdS stands for the ‘Einstein de-Sitter’ Universe which is a flat, matter dominated universe. Therefore, a growing solution becomes

$$D_+(a) = \frac{5}{2} \Omega_m \frac{H(a)}{H_0} \int_0^a \frac{da'}{[a' H(a')/H_0]^3}, \quad (2.10)$$

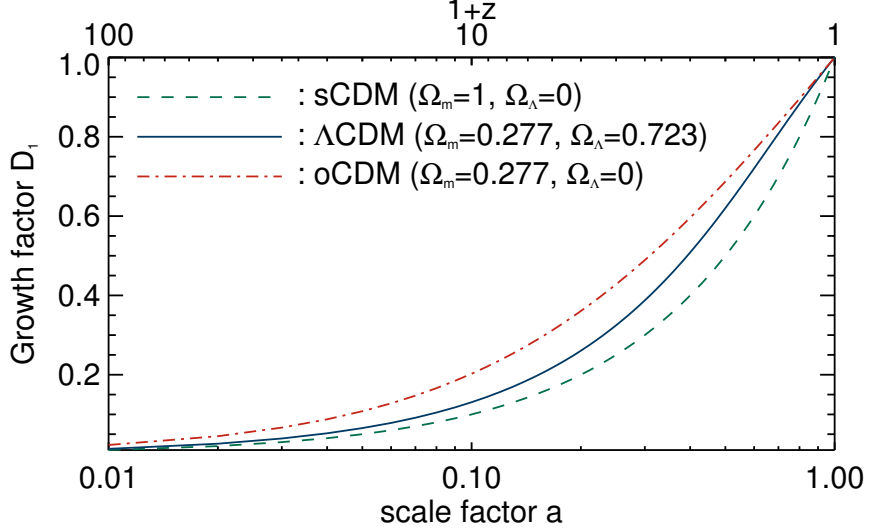


Figure 2.1: The linear growth factor, $D(a)$, for three different cosmologies: sCDM ($\Omega_m = 1$, $\Omega_\Lambda = 0$) Λ CDM ($\Omega_m = 0.277$, $\Omega_\Lambda = 0.723$) oCDM ($\Omega_m = 0.277$, $\Omega_\Lambda = 0$)

where Ω_m takes its present value. Another convention is normalizing its value to be unity at present:

$$D(a) = \frac{D_+(a)}{D_+(a=1)}. \quad (2.11)$$

Throughout this dissertation, we use the later convention, and call $D(a)$ the ‘linear growth factor’. Note that the two different conventions differ by a factor of 0.765 for the cosmological parameters in Table 1 (“WMAP+BAO+SN”) of Komatsu et al. (2009).

Figure 2.1 shows the linear growth factor for three different cosmologies: standard Cold Dark Matter (sCDM) model ($\Omega_m = 1$), Cold Dark Matter with cosmological constant (Λ CDM) model ($\Omega_m = 0.277$, $\Omega_\Lambda = 0.723$), and open Cold Dark Matter (oCDM) model ($\Omega_m = 0.277$, $\Omega_\Lambda = 0$). For given density fluctuations today, at high redshifts, the density fluctuations have to be larger for the oCDM universe, and smaller for sCDM universe compare to the standard Λ CDM universe. It is because in Λ CDM and oCDM universe, energy density is dominated by dark energy and curvature, respectively; both of them retard the growth of structure by speeding up the expansion of universe faster than the sCDM universe.

We calculate the velocity gradient field $\theta_1(\mathbf{k}, \tau)$ as

$$\theta_1(\mathbf{k}, \tau) = -\frac{\partial \delta_1(\mathbf{k}, \tau)}{\partial \tau} = -\frac{\delta_1(\mathbf{k}, \tau)}{D(\tau)} \frac{dD(\tau)}{d\tau} = -f(\tau)\mathcal{H}(\tau)\delta_1(\mathbf{k}, \tau), \quad (2.12)$$

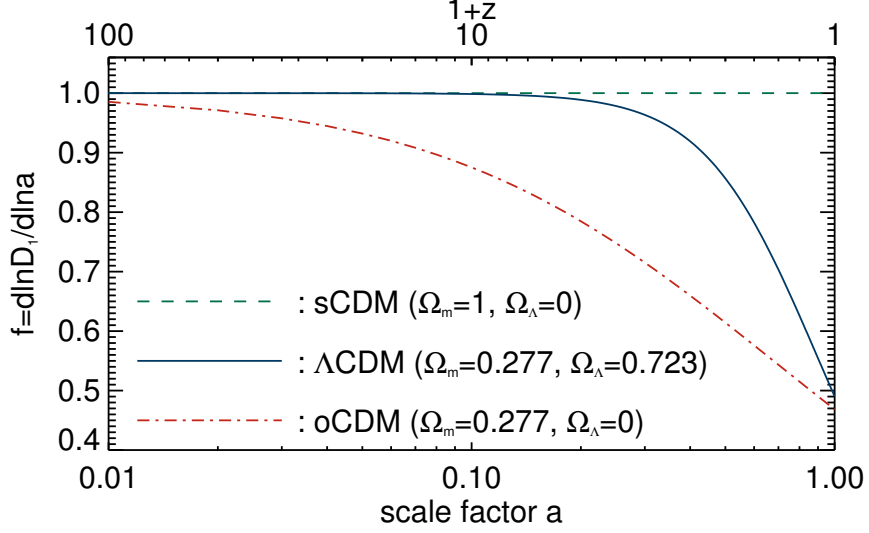


Figure 2.2: The logarithmic derivative of the linear growth factor, $f(a) \equiv d \ln D / d \ln a$, for three different cosmologies: sCDM ($\Omega_m = 1, \Omega_\Lambda = 0$) Λ CDM ($\Omega_m = 0.277, \Omega_\Lambda = 0.723$) oCDM ($\Omega_m = 0.277, \Omega_\Lambda = 0$)

where

$$f(\tau) \equiv \frac{d \ln D}{d \ln a} = \frac{1}{2} \left(\frac{H_0}{aH(a)} \right)^2 \left[\frac{5\Omega_m}{D_+(a)} - \frac{3\Omega_m}{a} - 2(1 - \Omega_m - \Omega_\Lambda) \right], \quad (2.13)$$

is the logarithmic growth rate. Here, Ω_m, Ω_Λ are the values at present.

Figure 2.2 shows the logarithmic growth rate for three different cosmologies: sCDM model, Λ CDM model, and oCDM model. When universe is flat, matter dominated, $f = 1$, and linear growth is slowing down once cosmological constant start to affect the expansion in Λ CDM universe. For oCDM universe, the growth rate is always slower than Λ CDM or sCDM universe.

What about the wave vector $\mathbf{k}$ dependence? We can divided the k -dependence of linear perturbation by two parts: k -dependence due to the generation of primordial perturbation from inflation, and k -dependence due to the subsequent evolution of density perturbation to matter epoch.

Inflation stretches the quantum fluctuation outside of horizon, and generate the primordial curvature perturbation, $\zeta(\mathbf{k})$, which is conserved outside of horizon even if the equation of state $w \equiv P/\rho$ changing (Mukhanov et al., 1992). The Bardeen's potential, a

relativistic generalization of the peculiar gravitational potential¹, $\Phi_H(\mathbf{k})$ is also conserved outside of horizon, but only for constant w , and for the universe dominated by a perfect fluid whose equation of state is w , it is related to the primordial curvature perturbation $\zeta(\mathbf{k})$ by

$$\Phi_H(\mathbf{k}) = \frac{3+3w}{5+3w}\zeta(\mathbf{k}). \quad (2.14)$$

When universe is dominated by radiation or matter, expansion of the universe decelerates, and the wavemodes once stretched outside of horizon by inflation start to re-enter inside of horizon. As $w = 0$ for matter, the Bardeen's potential of the mode which re-enter the horizon during matter era is $\Phi_H(\mathbf{k}) = 3/5\zeta(\mathbf{k})$ at horizon crossing time. Inside of horizon, the Bardeen's potential $\Phi(\mathbf{k}, a)$ is related to the density field² by the Poisson equation:

$$k^2\Phi(\mathbf{k}, a) = 4\pi G a^2 \bar{\rho}(a)\delta_1(\mathbf{k}, a) = \frac{3}{2}H_0^2\Omega_m(1+z)\delta_1(\mathbf{k}, a). \quad (2.15)$$

We denote the time evolution of the peculiar gravitational potential as $g(z)$, and it is apparent from equation (2.15) that

$$g(z) = (1+z)D(z). \quad (2.16)$$

Then, we rewrite the Bardeen's potential at large scales as

$$\Phi(\mathbf{k}, a) = g(z)\Phi(\mathbf{k}), \quad (2.17)$$

where $\Phi(\mathbf{k})$ is the Bardeen's potential extrapolated at present epoch³; thus, it is related to the horizon crossing value as $\Phi(\mathbf{k})/\Phi_H(\mathbf{k}) = 1/g(z)|_{\text{EdS}} = D_+(a=1) \simeq 0.765$. The numerical value is for cosmological parameters in Table 1 ("WMAP5+BAO+SN") of Komatsu et al. (2009).

On the other hand, for the wave modes re-enter horizon during radiation era, as perturbation of dominant component (radiation) cannot grow due to its pressure, peculiar gravitational potential decays and matter density contrast can only grow logarithmically. Therefore, the amplitude of sub-horizon perturbations are suppressed relative to the super-horizon perturbations. Plus, at that time baryons were tightly coupled to photon, and could not contribute to the growth of matter fluctuation.

¹Note that Φ_H has an opposite sign of the Newtonian peculiar gravitational potential ϕ we defined earlier.

²To be precise, this equation holds for *comoving gauge* where $\delta u = 0$.

³Throughout this dissertation, we consistently follow this convention: a dynamical quantity, such as Φ , $\delta_1 P_L$, written without explicit time (redshift) dependence denotes the quantity extrapolated to its present value.

In order to take into account these evolution, we need to solve the perturbed Einstein equation and Boltzmann equation for coupled dark matter, photon, baryon, neutrino system. There are many publically available code for calculating such equation systems; among them, CAMB⁴ and CMBFAST⁵ are most widely used in the cosmology community.

These codes calculate so called the ‘transfer function’ $T(k)$. The transfer function $T(k)$ encodes the evolution of density perturbation throughout the matter-radiation equality and CMB last scattering. Since transfer function is defined as the relative changes of small scale modes (which enter horizon earlier) compared to the large scale modes (which enter horizon during matter dominated epoch), the transfer function is unity on large scales: $T(k) = 1$. Therefore, the effects of the retarded growth in the radiation epoch and tight coupling between baryon-photon can be taken into account by multiplying the transfer function to the left hand side of equation (2.15):

$$\delta_1(\mathbf{k}, z) = \frac{2}{3} \frac{k^2 T(k)}{H_0^2 \Omega_m} D(z) \Phi(\mathbf{k}) \equiv \mathcal{M}(k) D(z) \Phi(\mathbf{k}). \quad (2.18)$$

Primordial curvature perturbation predicted by the most inflationary models, and confirmed by observations such as *WMAP* and *SDSS*, is characterized by *nearly* a scale invariant power spectrum. Therefore, we conventionally parametrize the shape of the primordial curvature power spectrum as

$$P_\zeta(k) = 2\pi^2 \Delta_{\mathcal{R}}^2(k_p) \left(\frac{k}{k_p} \right)^{n_s(k_p) - 4 + \frac{1}{2} \alpha_s \ln\left(\frac{k}{k_p}\right)}, \quad (2.19)$$

where we use three parameters: amplitude of primordial power spectrum $\Delta_{\mathcal{R}}^2$, spectral tilt n_s , and running index α_s . Here, k_p is a pivot wavenumber⁶. Note that the perfectly scale invariant primordial perturbation corresponds to $n_s = 1$, $\alpha_s = 0$.

Combining the primordial power spectrum [Eq (2.19)] and the late time linear evolution [Eq (2.18)], we calculate the linear matter power spectrum as

$$P_L(k) = \frac{8\pi^2}{25} \frac{[D_+(a=1)]^2}{H_0^4 \Omega_m^2} \Delta_{\mathcal{R}}^2(k_p) D^2(z) T^2(k) \left(\frac{k}{k_p} \right)^{n_s(k_p) + \frac{1}{2} \alpha_s \ln\left(\frac{k}{k_p}\right)}. \quad (2.20)$$

⁴<http://camb.info>

⁵<http://www.cmbfast.org>

⁶Different authors, surveys use different value of k_p . Komatsu et al. (2009) uses $k_p \equiv 0.002 \text{ [Mpc}^{-1}]$ for *WMAP*, while Reid et al. (2010) uses $k_p \equiv 0.05 \text{ [Mpc}^{-1}]$ for *SDSS*.

Alternatively, we can also normalize the linear power spectrum by fixing σ_8 , a r.m.s. density fluctuation smoothed by the spherical top-hat filter of radius 8 Mpc/ h , whose explicit formula is given by

$$\sigma_8^2 \equiv \int d \ln k \frac{k^3 P_L(k)}{2\pi^2} W^2(kR), \quad (2.21)$$

where

$$W(kR) = 3 \left[\frac{\sin(kR)}{k^3 R^3} - \frac{\cos(kR)}{k^2 R^2} \right]$$

with $R = 8 \text{ Mpc}/h$.

2.1.2 Non-linear solution for density field and velocity field

Let us come back to the original non-linear equations. In order to solve these coupled integro-differential equations, we shall expand $\delta_{\mathbf{k}}(\tau)$ and $\theta_{\mathbf{k}}(\tau)$ perturbatively by using the n -th power of linear density contrast $\delta_1(\mathbf{k}, \tau)$ as a basis:

$$\begin{aligned} \delta_{\mathbf{k}}(\tau) &= \sum_{n=1}^{\infty} \int \frac{d^3 q_1}{(2\pi)^3} \cdots \int \frac{d^3 q_{n-1}}{(2\pi)^3} \int d^3 q_n \delta^D(\mathbf{k} - \sum_{i=1}^n \mathbf{q}_i) \\ &\quad \times F_n^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n, \tau) \delta_1(\mathbf{q}_1, \tau) \cdots \delta_1(\mathbf{q}_n, \tau) \end{aligned} \quad (2.22)$$

$$\begin{aligned} \theta_{\mathbf{k}}(\tau) &= -f(\tau) \mathcal{H}(\tau) \sum_{n=1}^{\infty} \int \frac{d^3 q_1}{(2\pi)^3} \cdots \int \frac{d^3 q_{n-1}}{(2\pi)^3} \int d^3 q_n \delta^D(\mathbf{k} - \sum_{i=1}^n \mathbf{q}_i) \\ &\quad \times G_n^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n, \tau) \delta_1(\mathbf{q}_1, \tau) \cdots \delta_1(\mathbf{q}_n, \tau). \end{aligned} \quad (2.23)$$

Here, $f(\tau) = d \ln D / d \ln a$ with the linear growth factor $D(a)$, and $F_n^{(s)}$ and $G_n^{(s)}$ are symmetrized kernels, which characterize coupling between different wave modes. We introduce $f(\tau) \mathcal{H}(\tau)$ factor in equation (2.23) motivated by the linear relation between density contrast and velocity divergence: $\theta_1(\mathbf{k}, \tau) = -f(\tau) \mathcal{H}(\tau) \delta_1(\mathbf{k}, \tau)$. In this definition the first order kernels become unity: $F_1^{(s)} = G_1^{(s)} = 1$.

As we know the evolution of $\delta_1(\mathbf{q}, \tau)$ from the linear theory, calculating $F_n^{(s)}$ and $G_n^{(s)}$ will complete the solution. The standard procedure (Goroff et al., 1986; Jain & Bertschinger, 1994) is calculating the *un-symmetized* kernels F_n and G_n first, then symmetrize them under changing arguments:

$$F_n^{(s)}(\mathbf{q}_1, \cdots, \mathbf{q}_n) = \frac{1}{n!} \sum_{\sigma} F_n(\mathbf{q}_{\sigma_1}, \cdots, \mathbf{q}_{\sigma_n}) \quad (2.24)$$

$$G_n^{(s)}(\mathbf{q}_1, \cdots, \mathbf{q}_n) = \frac{1}{n!} \sum_{\sigma} G_n(\mathbf{q}_{\sigma_1}, \cdots, \mathbf{q}_{\sigma_n}). \quad (2.25)$$

Here, sum is taken for all the permutations $\sigma \equiv (\sigma_1, \dots, \sigma_n)$ of the set $\{1, \dots, n\}$. By substituting the perturbative expansion of equation (2.22) and equation (2.23) back into the original equations, we get the equations of $F_n^{(s)}$ s and $G_n^{(s)}$ s. For example, the continuity equation [Eq. (2.4)] becomes

$$\begin{aligned}
& \sum_{n=1}^{\infty} \int \frac{d^3 q_1}{(2\pi)^3} \cdots \int \frac{d^3 q_{n-1}}{(2\pi)^3} \int d^3 q_n \delta^D(\mathbf{k} - \sum_{i=1}^n \mathbf{q}_i) \delta_1(\mathbf{q}_1, \tau) \cdots \delta_1(\mathbf{q}_n, \tau) \\
& \times \left[\frac{\partial F_n^{(s)}(\mathbf{q}_1, \dots, \mathbf{q}_n, \tau)}{\partial \tau} + n \mathcal{H} f F_n^{(s)}(\mathbf{q}_1, \dots, \mathbf{q}_n, \tau) - \mathcal{H} f G_n^{(s)}(\mathbf{q}_1, \dots, \mathbf{q}_n, \tau) \right] \\
& = (2\pi)^3 \mathcal{H} f \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \int \frac{d^3 q_{11}}{(2\pi)^3} \cdots \int \frac{d^3 q_{1m}}{(2\pi)^3} \int \frac{d^3 q_{21}}{(2\pi)^3} \cdots \int \frac{d^3 q_{2l}}{(2\pi)^3} \\
& \times \frac{\mathbf{k} \cdot (\mathbf{q}_{11} + \cdots + \mathbf{q}_{1m})}{|\mathbf{q}_{11} + \cdots + \mathbf{q}_{1m}|^2} G_m^{(s)}(\mathbf{q}_{11}, \dots, \mathbf{q}_{1m}) F_l^{(s)}(\mathbf{q}_{21}, \dots, \mathbf{q}_{2l}) \\
& \times \delta^D(\mathbf{k} - \sum_{i=1}^m \mathbf{q}_{1i} - \sum_{j=1}^l \mathbf{q}_{2j}) \delta_{11}(\mathbf{q}_1, \tau) \cdots \delta_{1m}(\mathbf{q}_m, \tau) \delta_{21}(\mathbf{q}_1, \tau) \cdots \delta_{2l}(\mathbf{q}_l, \tau).
\end{aligned}$$

Here, we use an identity from linear theory $\partial \delta_1 / \partial \tau = f \mathcal{H} \delta_1$ to replace the conformal-time (τ) derivatives. In order to isolate the equation for the kernels, we have to identify $\mathbf{q}_i$ s to $\mathbf{q}_{1i}$ s and $\mathbf{q}_{2i}$ s. In principle, one can take all possible identifications and symmetrize the equation itself, but we follow the standard approach where (for given n) indexes are matched as $(\mathbf{q}_{11}, \dots, \mathbf{q}_{1m}) = (\mathbf{q}_1, \dots, \mathbf{q}_m)$, and $(\mathbf{q}_{21}, \dots, \mathbf{q}_{2l}) = (\mathbf{q}_{m+1}, \dots, \mathbf{q}_n)$. By matching indexes this way, the equations are not manifestly symmetric anymore, and as a result the kernels are not symmetric: we drop the superscript (s) and symmetrize kernels later by using equation (2.24) and equation (2.25).

Once $\mathbf{q}_i$ s are identified, we can read off the equation for F_n and G_n from the equation above as

$$\begin{aligned}
& \frac{1}{f(\tau) \mathcal{H}(\tau)} \frac{\partial F_n(\mathbf{k}, \tau)}{\partial \tau} + n F_n(\mathbf{k}, \tau) - G_n(\mathbf{k}, \tau) \\
& = \sum_{m=1}^{n-1} \frac{\mathbf{k} \cdot \mathbf{k}_1}{k_1^2} G_m(\mathbf{k}_1, \tau) F_{n-m}(\mathbf{k}_2, \tau),
\end{aligned} \tag{2.26}$$

where we use the short hand notation of $F_n(\mathbf{k}, \tau) \equiv F_n(\mathbf{q}_1, \dots, \mathbf{q}_n, \tau)$, and $G_n(\mathbf{k}, \tau) \equiv G_n(\mathbf{q}_1, \dots, \mathbf{q}_n, \tau)$ with a constraint $\mathbf{k} = \mathbf{q}_1 + \cdots + \mathbf{q}_n$. Note that from the matching condition of $\mathbf{q}_i$ s, $G_m(\mathbf{k}_1, \tau) = G_n(\mathbf{q}_1, \dots, \mathbf{q}_m, \tau)$ and $F_{n-m}(\mathbf{k}_2, \tau) = F_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n, \tau)$ in the right hand side of equation (2.26). Also, Dirac delta function dictates $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}$. Similarly,

we find

$$\begin{aligned} & \frac{1}{f(\tau)\mathcal{H}(\tau)} \frac{\partial G_n(\mathbf{k}, \tau)}{\partial \tau} + \left[\frac{3}{2} \frac{\Omega_m(\tau)}{f^2(\tau)} + n - 1 \right] G_n(\mathbf{k}, \tau) - \frac{3}{2} \frac{\Omega_m(\tau)}{f^2(\tau)} F_n(\mathbf{k}, \tau) \\ &= \sum_{m=1}^{n-1} \frac{k^2(\mathbf{k}_1 \cdot \mathbf{k}_2)}{2k_1^2 k_2^2} G_m(\mathbf{k}_1, \tau) G_{n-m}(\mathbf{k}_2, \tau), \end{aligned} \quad (2.27)$$

from the Euler equation [Eq. (2.5)].

In general, the kernels F_n and G_n depend on time, and we have to solve complicated differential equations, [Eq. (2.26)] and [Eq. (2.27)]. However, it is well known that the kernels are extremely insensitive to the underlying cosmology, and the next-to-leading order correction to $P(k)$ can be correctly modeled as long as one uses the correct growth factor for $\delta_1(\mathbf{k}, \tau)$ (Bernardeau et al., 2002). Therefore, we shall calculate the kernels in the Einstein de-Sitter (spatially flat, matter-dominated) universe. In Einstein de-Sitter Universe, as $\Omega_m/f^2 = 1$, F_n and G_n are constant in time and equation (2.26) and equation (2.27) reduce to the algebraic equations. Moreover, Takahashi (2008) has calculated the exact solution up to third order in general dark energy model, and has concluded that the difference between the next-to-leading order power spectrum from exact kernels and that from Einstein de-Sitter kernels is extremely small. It is at most sub-percent level at $z = 0$, and decreases as redshift increases to $\sim 10^{-4}$ at $z = 3$. As we are mostly interested in the high redshift ($z > 1$), we can safely ignore such a small difference in the kernels.

In Einstein de-Sitter Universe, equation (2.26) and equation (2.27) become

$$nF_n(\mathbf{k}) - G_n(\mathbf{k}) = \sum_{m=1}^{n-1} \frac{\mathbf{k} \cdot \mathbf{k}_1}{k_1^2} G_m(\mathbf{k}_1) F_{n-m}(\mathbf{k}_2), \quad (2.28)$$

$$(2n+1)G_n(\mathbf{k}) - 3F_n(\mathbf{k}) = \sum_{m=1}^{n-1} \frac{k^2(\mathbf{k}_1 \cdot \mathbf{k}_2)}{k_1^2 k_2^2} G_m(\mathbf{k}_1) G_{n-m}(\mathbf{k}_2). \quad (2.29)$$

By solving the algebraic equations, we find the recursion relations

$$\begin{aligned} F_n(\mathbf{q}_1, \dots, \mathbf{q}_n) &= \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \dots, \mathbf{q}_m)}{(2n+3)(n-1)} \left\{ (2n+1) \frac{\mathbf{k} \cdot \mathbf{k}_1}{k_1^2} F_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \right. \\ &\quad \left. + \frac{k^2(\mathbf{k}_1 \cdot \mathbf{k}_2)}{k_1^2 k_2^2} G_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \right\} \end{aligned} \quad (2.30)$$

$$\begin{aligned} G_n(\mathbf{q}_1, \dots, \mathbf{q}_n) &= \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \dots, \mathbf{q}_m)}{(2n+3)(n-1)} \left\{ 3 \frac{\mathbf{k} \cdot \mathbf{k}_1}{k_1^2} F_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \right. \\ &\quad \left. + n \frac{k^2(\mathbf{k}_1 \cdot \mathbf{k}_2)}{k_1^2 k_2^2} G_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \right\}. \end{aligned} \quad (2.31)$$

With the recursion relations and the kernels for linear theory, $F_1 = G_1 = 1$, we can calculate $F_n^{(s)}, G_n^{(s)}$ for all order. For example, the second order kernels are

$$F_2^{(s)}(\mathbf{k}_1, \mathbf{k}_2) = \frac{5}{7} + \frac{2}{7} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} + \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{2} \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} \right), \quad (2.32)$$

$$G_2^{(s)}(\mathbf{k}_1, \mathbf{k}_2) = \frac{3}{7} + \frac{4}{7} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} + \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{2} \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} \right), \quad (2.33)$$

and third order kernels can be calculated from second order kernels as⁷

$$\begin{aligned} F_3^{(s)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = & \frac{2k^2}{54} \left[\frac{\mathbf{k}_1 \cdot \mathbf{k}_{23}}{k_1^2 k_{23}^2} G_2^{(s)}(\mathbf{k}_2, \mathbf{k}_3) + (2 \text{ cyclic}) \right] \\ & + \frac{7}{54} \mathbf{k} \cdot \left[\frac{\mathbf{k}_{12}}{k_{12}^2} G_2^{(s)}(\mathbf{k}_1, \mathbf{k}_2) + (2 \text{ cyclic}) \right] \\ & + \frac{7}{54} \mathbf{k} \cdot \left[\frac{\mathbf{k}_1}{k_1^2} F_2^{(s)}(\mathbf{k}_2, \mathbf{k}_3) + (2 \text{ cyclic}) \right], \end{aligned} \quad (2.34)$$

and

$$\begin{aligned} G_3^{(s)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = & \frac{k^2}{9} \left[\frac{\mathbf{k}_1 \cdot \mathbf{k}_{23}}{k_1^2 k_{23}^2} G_2^{(s)}(\mathbf{k}_2, \mathbf{k}_3) + (2 \text{ cyclic}) \right] \\ & + \frac{1}{18} \mathbf{k} \cdot \left[\frac{\mathbf{k}_{12}}{k_{12}^2} G_2^{(s)}(\mathbf{k}_1, \mathbf{k}_2) + (2 \text{ cyclic}) \right] \\ & + \frac{1}{18} \mathbf{k} \cdot \left[\frac{\mathbf{k}_1}{k_1^2} F_2^{(s)}(\mathbf{k}_2, \mathbf{k}_3) + (2 \text{ cyclic}) \right]. \end{aligned} \quad (2.35)$$

In summary, the solution of Eulerian Perturbation Theory consists of the perturbative expansions (Eq. (2.22) and Eq. (2.23)) with kernels calculated by recursion relations (Eq. (2.30) and Eq. (2.31)) followed by symmetrization (Eq. (2.24) and Eq. (2.25)). By using this solution, we can describe the non-linear growth of density field and velocity field of cosmic matter in the quasi-nonlinear regime.

2.2 Statistics of the cosmological density field: Gaussian vs. non-Gaussian

In the previous section, we calculate the solution for the non-linear evolution of the density and velocity fields in terms of the linear density field. As the direct observable of the

⁷For the compactness of the equation, we adopt the short-hand notation of

$$\mathbf{k}_{ijk\dots} \equiv \mathbf{k}_i + \mathbf{k}_j + \mathbf{k} + \dots$$

throughout this dissertation.

cosmological observation is the statistical correlation function such as power spectrum and bispectrum, in order to compare the non-linear solution to observation, we have to calculate the statistical correlation of density and velocity field.

Inflationary theories predict that the primordial curvature perturbation obeys a *nearly* Gaussian statistics; thus, the linear density field $\delta_1(\mathbf{k})$, which evolves linearly from the curvature perturbation [Eq. (2.18)], also obeys the same statistics. Let us first consider the Gaussian statistics.

In order to describe the statistics of a field, we have to introduce a *probability functional* $\mathcal{P}[\delta_1(\mathbf{x})]$, which describes the probability of having a configuration of density field δ_1 whose value is $\delta_1(\mathbf{x})$ at a point $\mathbf{x}$. Note that statistical homogeneity dictates $\mathcal{P}[\delta_1(\mathbf{x})]$ to be independent of position $\mathbf{x}$. For Gaussian case, it is given by (Gabrielli et al., 2005)

$$\mathcal{P}[\delta_1(\mathbf{x})] = \frac{1}{Z} \exp \left[-\frac{1}{2} \int d^3y \int d^3z \delta_1(\mathbf{y}) K(\mathbf{y}, \mathbf{z}) \delta_1(\mathbf{z}) \right], \quad (2.36)$$

where $K(\mathbf{y}, \mathbf{z})$ is corresponding to the inverse of the covariance matrix C^{-1} in the usual Gaussian statistics of discrete variables, and

$$Z = \int [\mathcal{D}\delta_1] \exp \left[-\frac{1}{2} \int d^3y \int d^3z \delta_1(\mathbf{y}) K(\mathbf{y}, \mathbf{z}) \delta_1(\mathbf{z}) \right] \quad (2.37)$$

is the normalization constant, and $[\mathcal{D}\delta_1]$ is the integration measure in the Hilbert space.

By using the probability functional, we calculate the n -point correlation function of $\delta_1(\mathbf{x})$ by taking the expectation value as:

$$\langle \delta_1(\mathbf{x}_1) \cdots \delta_1(\mathbf{x}_n) \rangle = \int [\mathcal{D}\delta_1(\mathbf{x})] \delta_1(\mathbf{x}_1) \cdots \delta_1(\mathbf{x}_n) \mathcal{P}[\delta_1(\mathbf{x})]. \quad (2.38)$$

A usual technique of calculating the expectation value is by introducing the *generating functional*

$$Z[J] \equiv \left\langle e^{i \int d^3x J(\mathbf{x}) \delta_1(\mathbf{x})} \right\rangle, \quad (2.39)$$

and taking successive functional derivatives with respect to $J(\mathbf{x}_i)$. Finally, we get the correlation function by setting $J = 0$. That is, n -point correlation function is

$$\langle \delta_1(\mathbf{x}_1) \cdots \delta_1(\mathbf{x}_n) \rangle = \frac{1}{i^n} \frac{\delta^n Z[J]}{\delta J(\mathbf{x}_1) \cdots \delta J(\mathbf{x}_n)} \Big|_{J=0}. \quad (2.40)$$

Furthermore, we can also calculate the n -point *connected* correlation function by

$$\langle \delta_1(\mathbf{x}_1) \cdots \delta_1(\mathbf{x}_n) \rangle_c = \frac{1}{i^n} \frac{\delta^n \ln Z[J]}{\delta J(\mathbf{x}_1) \cdots \delta J(\mathbf{x}_n)} \Big|_{J=0}. \quad (2.41)$$

For Gaussian case, we can re-write the generating functional in a closed form:

$$Z[J] = \frac{1}{Z} \int [\mathcal{D}f] \exp \left[-\frac{1}{2} \int d^3y \int d^3z f(\mathbf{y}) K(\mathbf{y}, \mathbf{z}) f(\mathbf{z}) + i \int d^3x J(\mathbf{x}) f(\mathbf{x}) \right]. \quad (2.42)$$

Then, we make the coordinate transformation of

$$\delta'_1(\mathbf{x}) = \delta_1(\mathbf{x}) - i \int d^3y K^{-1}(\mathbf{x}, \mathbf{y}) J(\mathbf{y}),$$

where $K^{-1}(\mathbf{x}, \mathbf{y})$ is the functional inverse of $K(\mathbf{x}, \mathbf{y})$, which is defined by

$$\int d^3z K(\mathbf{x}, \mathbf{z}) K^{-1}(\mathbf{z}, \mathbf{y}) = \delta^D(\mathbf{x} - \mathbf{y}).$$

This coordinate transformation does not change the integral measure, as is simple translation in the functional space. Under the coordinate transformation, exponent in equation (2.42) becomes

$$-\frac{1}{2} \int d^3y \int d^3z \delta'_1(\mathbf{y}) K(\mathbf{y}, \mathbf{z}) \delta'_1(\mathbf{z}) - \frac{1}{2} \int d^3x \int d^3y J(\mathbf{x}) K^{-1}(\mathbf{x}, \mathbf{y}) J(\mathbf{y}), \quad (2.43)$$

where the functional integration of the first term is unity; thus, generating functional for Gaussian case becomes

$$Z[J] = \exp \left[-\frac{1}{2} \int d^3x \int d^3y J(\mathbf{x}) K^{-1}(\mathbf{x}, \mathbf{y}) J(\mathbf{y}) \right]. \quad (2.44)$$

Therefore, by using equation (2.41), we calculate the n -point correlation function. By taking the functional derivative twice, we find that

$$\begin{aligned} \frac{\delta Z[J]}{\delta J(\mathbf{x}_1)} &= -\frac{1}{2} \left[\int d^3x \int d^3y \delta^D(\mathbf{x} - \mathbf{x}_1) K^{-1}(\mathbf{x}, \mathbf{y}) J(\mathbf{y}) \right. \\ &\quad \left. + \int d^3x \int d^3y J(\mathbf{x}) K^{-1}(\mathbf{x}, \mathbf{y}) \delta^D(\mathbf{y} - \mathbf{x}_1) \right] Z[J] \\ &= - \left[\int d^3x K^{-1}(\mathbf{x}, \mathbf{x}_1) J(\mathbf{x}) \right] Z[J], \end{aligned}$$

and

$$\frac{\delta^2 Z[J]}{\delta J(\mathbf{x}_1) \delta J(\mathbf{x}_2)} = -K^{-1}(\mathbf{x}, \mathbf{x}_1) Z[J].$$

By carefully observing this procedure, we find that any odd-number n -point correlation function has to vanish when setting $J = 0$, and any even-number n -point correlation function is given only by $K^{-1}(\mathbf{x}, \mathbf{y})$. Especially, from the calculation above, we can read that the two point correlation function is given by

$$\xi(\mathbf{x}_1, \mathbf{x}_2) \equiv \langle \delta_1(\mathbf{x}_1) \delta_1(\mathbf{x}_2) \rangle = K^{-1}(\mathbf{x}_1, \mathbf{x}_2). \quad (2.45)$$

This result is so called *Wick's theorem*, which states that “ the average of a product of an even number of δ_1 s is the sum over all ways of pairing δ_1 s with each other of a product of the average values of the pairs:

$$\langle \delta_1(\mathbf{x}_1)\delta_1(\mathbf{x}_2)\cdots \rangle = \sum_{\text{pairings}} \prod_{\text{pairs}} \langle \delta_1\delta_1 \rangle, \quad (2.46)$$

with the sum over pairings not distinguishing those which interchange coordinates in a pair, or which merely interchange pairs” (Weinberg, 2008). What about the connected correlation function? From equation (2.41), it is clear that, for Gaussian case, all higher order ($n > 2$) connected correlation functions vanish:

$$\langle \delta_1(\mathbf{x}_1)\cdots\delta_1(\mathbf{x}_n) \rangle_c = -\frac{1}{2i^n} \frac{\delta^n}{\delta J(\mathbf{x}_1)\cdots\delta J(\mathbf{x}_n)} \int d^3x \int d^3y J(\mathbf{x}) K^{-1}(\mathbf{x}, \mathbf{y}) J(\mathbf{y}).$$

From equation (2.45), $K(\mathbf{x}, \mathbf{y})$ is related to the correlation function by

$$\int d^3z \xi(\mathbf{x}, \mathbf{z}) K(\mathbf{z}, \mathbf{y}) = \delta^D(\mathbf{x} - \mathbf{y}). \quad (2.47)$$

Let us further investigate on the kernel, $K(\mathbf{x}, \mathbf{y})$. Due to the statistical homogeneity and isotropy of universe, $\xi(\mathbf{x}, \mathbf{z})$ and $K(\mathbf{x}, \mathbf{y})$ have to depend only on the separation, i.e. $\xi(\mathbf{x}, \mathbf{z}) = \xi(\mathbf{x} - \mathbf{z})$. Therefore,

$$\int d^3z \xi_f(\mathbf{x} - \mathbf{z}) K(\mathbf{z} - \mathbf{y}) = \delta^D(\mathbf{x} - \mathbf{y}). \quad (2.48)$$

Then, Fourier transform of the equation above leads

$$K(\mathbf{q}) = \frac{1}{P_L(q)}, \quad (2.49)$$

where $P_L(q)$ is the linear power spectrum, which is related to the two-point correlation function by

$$\xi(r) = \int \frac{d^3q}{(2\pi)^3} P_L(q) e^{i\mathbf{q}\cdot\mathbf{r}}. \quad (2.50)$$

That is, $K(\mathbf{x} - \mathbf{y})$ is Fourier transform of the inverse of linear power spectrum $P_L(q)$. One can also show that directly by using Fourier space representation of the linear density field $\delta_1(\mathbf{k})$. By Fourier transforming the exponent of equation (2.36), we find

$$\begin{aligned} & -\frac{1}{2} \int d^3y \int d^3z \delta_1(\mathbf{y}) K(\mathbf{y} - \mathbf{z}) \delta_1(\mathbf{z}) \\ &= -\frac{1}{2} \int d^3q_1 \int d^3q_2 \delta_1(\mathbf{q}_1) \left[\frac{K(\mathbf{q}_1)}{(2\pi)^3} \delta^D(\mathbf{q}_1 + \mathbf{q}_2) \right] \delta_1(\mathbf{q}_2). \end{aligned} \quad (2.51)$$

As the structure of the probability functional is the same as that for the real space, we can simply relate $K(\mathbf{q})$ to the power spectrum by using the analogy. Because the functional inversion of the Fourier space kernel is

$$\left[\frac{K(\mathbf{q}_1)\delta^D(\mathbf{q}_1 + \mathbf{q}_2)}{(2\pi)^3} \right]^{-1} = \frac{(2\pi)^3}{K(\mathbf{q}_1)}\delta^D(\mathbf{q}_1 + \mathbf{q}_2), \quad (2.52)$$

the power spectrum is given by

$$\langle \delta_1(\mathbf{q}_1)\delta_1(\mathbf{q}_2) \rangle \equiv (2\pi)^3 P(q_1)\delta^D(\mathbf{q}_1 + \mathbf{q}_2) = \frac{(2\pi)^3}{K(\mathbf{q}_1)}\delta^D(\mathbf{q}_1 + \mathbf{q}_2). \quad (2.53)$$

By using the second equality, we reproduce equation (2.49). Politzer & Wise (1984) extended this method to calculate the n -point correlation function of peaks.

As we have shown here, when the linear density field follows the Gaussian statistics, all higher order statistical properties are solely determined by the two point correlation function, or power spectrum. For non-Gaussian case, however, the connected n -point correlation functions do not vanish, and we have to know the probability functional or generating functional in order to calculate the statistical quantities. Conversely, if we know the n -point correlation functions to infinite order, then, we can calculate the generating functional by inverting equation (2.41):

$$\ln Z[J] = \sum_{n=2}^{\infty} \frac{i^n}{n!} \int d^3y_1 \cdots \int d^3y_n \langle \delta_1(\mathbf{y}_1) \cdots \delta_1(\mathbf{y}_n) \rangle_c J(\mathbf{y}_1) \cdots J(\mathbf{y}_n), \quad (2.54)$$

thus fully specify the statistics of the linear density field. Matarrese et al. (1986) derives the n -point correlation function of peaks by using this method (MLB formula), and we shall calculate the galaxy bispectrum in the presence of primordial non-Gaussianity in Chapter 5 based on this formula. The full derivation of MLB formula is also shown in Appendix K.

2.3 Next-to-leading order power spectrum from the perturbative solution: the theoretical template

In the following sections, we shall calculate the power spectrum of density contrast of matter in real space (Section 2.4), matter in redshift space (Section 2.6), galaxies in real space (Section 2.5), and galaxies in redshift space (Section 2.7). For all cases, we shall calculate corresponding density contrasts as a function of the matter density contrast $\delta_{\mathbf{k}}(\tau)$ and velocity gradient $\theta_{\mathbf{k}}(\tau)$ for which we know the perturbation theory solution.

Therefore, in most general form, we can also write a quantity $X(\mathbf{k}, \tau)$, of which we want to calculate power spectrum, as a sum of the perturbative series

$$\begin{aligned} X(\mathbf{k}, \tau) &= X^{(1)}(\mathbf{k}, \tau) + X^{(2)}(\mathbf{k}, \tau) + X^{(3)}(\mathbf{k}, \tau) + \dots \\ &= \sum_{n=1}^{\infty} \int \frac{d^3 q_1}{(2\pi)^3} \dots \int \frac{d^3 q_{n-1}}{(2\pi)^3} \int d^3 q_n \delta^D(\mathbf{k} - \sum_{i=1}^n \mathbf{q}_i) \\ &\quad \times K_n^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \delta_1(\mathbf{q}_1, \tau) \dots \delta_1(\mathbf{q}_n, \tau), \end{aligned} \quad (2.55)$$

with a general symmetrized kernel $K_n^{(s)}(\mathbf{q}_1, \dots, \mathbf{q}_n)$, which is given by a combination of $F_n^{(s)}$ and $G_n^{(s)}$. Here, $X^{(n)}(\mathbf{k})$ denote that the quantity is n -th order in linear density contrast, $\delta_1(\mathbf{k}, \tau)$. For a quantity which can be expanded as equation (2.55), we can calculate next-to-leading order (called *one-loop*) power spectrum and bispectrum as a function of linear order (called *tree-level*) quantities. In this section, we present these general formulas which we shall use in the rest of this chapter.

As the power spectrum, $P_X(\mathbf{k}, \tau)$, is a quadratic quantity of $X(\mathbf{k}, \tau)$ in Fourier space,

$$\langle X(\mathbf{k})X(\mathbf{k}') \rangle = (2\pi)^3 P_X(\mathbf{k}) \delta^D(\mathbf{k} + \mathbf{k}'), \quad (2.56)$$

for the expansion in equation (2.55), the left hand side of equation (2.56) becomes

$$\begin{aligned} &\langle X(\mathbf{k}, \tau)X(\mathbf{k}', \tau) \rangle \\ &= \langle X^{(1)}(\mathbf{k}, \tau)X^{(1)}(\mathbf{k}', \tau) \rangle + 2\langle \text{Re} \left[X^{(1)}(\mathbf{k}, \tau)X^{(2)}(\mathbf{k}', \tau) \right] \rangle \\ &\quad + \langle X^{(2)}(\mathbf{k}, \tau)X^{(2)}(\mathbf{k}', \tau) \rangle + 2\langle \text{Re} \left[X^{(1)}(\mathbf{k}, \tau)X^{(3)}(\mathbf{k}', \tau) \right] \rangle + \dots \end{aligned} \quad (2.57)$$

Here, we use the Hermitianity (Reality) condition,

$$X^{(i)}(\mathbf{k}, \tau) = \left[X^{(i)}(-\mathbf{k}, \tau) \right]^* = \left[X^{(i)}(\mathbf{k}', \tau) \right]^*,$$

and Re takes the real part of a complex number. In equation (2.57), we explicitly write down the terms up to order $\mathcal{O}(\delta_1^4)$.

2.3.1 One-loop Power spectrum with Gaussian linear density field

When linear density contrast, δ_1 , is Gaussian, odd power of δ_1 vanishes when taking an ensemble average. Therefore, the next-to-leading order nonlinear power spectrum consists of $\langle X^{(1)}(\mathbf{k}, \tau)X^{(1)}(\mathbf{k}', \tau) \rangle$, $\langle X^{(2)}(\mathbf{k}, \tau)X^{(2)}(\mathbf{k}', \tau) \rangle$ and $\langle X^{(1)}(\mathbf{k}, \tau)X^{(3)}(\mathbf{k}', \tau) \rangle$ which is often

denoted as $P_{X,11}(\mathbf{k}, \tau)$, $P_{X,22}(\mathbf{k}, \tau)$ and $P_{X,13}(\mathbf{k}, \tau)$, respectively. We calculate the next-to-leading order power spectrum by using Wick's theorem as⁸

$$\begin{aligned}
& P_X(\mathbf{k}, \tau) \\
& \equiv P_{X,11}(\mathbf{k}, \tau) + P_{X,22}(\mathbf{k}, \tau) + 2P_{X,13}(\mathbf{k}, \tau) \\
& = \left[K_1^{(s)}(\mathbf{k}) \right]^2 P_L(k, \tau) + 2 \int \frac{d^3 q}{(2\pi)^3} P_L(q, \tau) P_L(|\mathbf{k} - \mathbf{q}|, \tau) \left[K_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \right]^2 \\
& \quad + 6K_1^{(s)}(\mathbf{k}) P_L(k, \tau) \int \frac{d^3 q}{(2\pi)^3} P_L(q, \tau) K_3^{(s)}(\mathbf{q}, -\mathbf{q}, \mathbf{k}), \tag{2.58}
\end{aligned}$$

where $P_L(k)$ is the linear power spectrum.

2.3.2 One-loop Power spectrum with non-Gaussian linear density field

When linear density contrast follows non-Gaussian distribution, the second term in equation (2.57), $\langle X^{(1)}(\mathbf{k}) X^{(2)}(\mathbf{k}') \rangle$, is non-zero, and becomes the leading order non-Gaussian correction to the power spectrum,

$$\begin{aligned}
\Delta P_{X,NG}(\mathbf{k}, \tau) &= 2P_{12}(\mathbf{k}, \tau) \\
&= 2K_1^{(s)}(\mathbf{k}) \int \frac{d^3 q}{(2\pi)^3} K_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) B_L(-\mathbf{k}, \mathbf{q}, \mathbf{k} - \mathbf{q}, \tau). \tag{2.59}
\end{aligned}$$

Here, B_L is the bispectrum of linear density field. There also are the non-Gaussian correction term coming from $P_{X,22}(\mathbf{k}, \tau)$ and $P_{X,13}(\mathbf{k}, \tau)$ proportional to the linear trispectrum, $T_L(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ ⁹:

$$\begin{aligned}
\Delta P_{X,22}(\mathbf{k}, \tau) &= \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} K_2^{(s)}(\mathbf{q}_1, \mathbf{k} - \mathbf{q}_1) K_2^{(s)}(\mathbf{q}_2, -\mathbf{k} - \mathbf{q}_2) \\
&\quad \times T_L(\mathbf{q}_1, \mathbf{k} - \mathbf{q}_1, \mathbf{q}_2, -\mathbf{k} - \mathbf{q}_2, \tau) \tag{2.60}
\end{aligned}$$

$$\begin{aligned}
\Delta P_{X,13}(\mathbf{k}, \tau) &= K_1^{(s)}(\mathbf{k}) \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} K_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) \\
&\quad \times T_L(-\mathbf{k}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2, \tau) \tag{2.61}
\end{aligned}$$

Therefore, in order to calculate the effect of primordial non-Gaussianity fully in one-loop level of perturbation theory, we need to know both linear bispectrum and the linear trispectrum. However, linear trispectrum ($T_L \sim \mathcal{M}^4 P_\phi^3$) is parametrically smaller

⁸For the derivation of $P_{X,22}$ and $P_{X,13}$, see Appendix B.1.

⁹For derivation, see Appendix B.2

than linear bispectrum ($B_L \sim \mathcal{M}^3 P_\phi^1$) in the quasi-nonlinear regime, as the primordial curvature power spectrum P_ϕ sharply decreases as k increases, $P_\phi(k) \sim k^{n_s-4}$. In other words, linear trispectrum arises from the two-loop contribution of the primordial curvature perturbation, we shall neglect $\Delta P_{X,22}$ and $\Delta P_{X,13}$ correction terms (Taruya et al., 2008). However, note that the trispectrum term may be important on the very large scales. Indeed $\Delta P_{X,22}$ term include a term proportional to $f_{\text{NL}}^2/(k^4 T^2(k))$ which sharply peaks on large scales (Desjacques & Seljak, 2010). In the rest of the chapter, we shall use equation (2.59) to calculate the non-Gaussian correction to the one-loop power spectrum.

2.3.2.1 Local primordial non-Gaussianity

Although we can apply equation (2.59) to any kind of non-Gaussianities, we shall focus on the local type non-Gaussianity. The local type non-Gaussianity is particularly interesting because of the consistency relation (Maldacena, 2003; Acquaviva et al., 2003; Creminelli & Zaldarriaga, 2004), which states that the coefficient of the squeezed bispectrum [Eq. (2.64)], f_{NL} , has to satisfy

$$f_{\text{NL}} = \frac{5}{12}(1 - n_s) \quad (2.62)$$

for *any* kind of single field inflation model. Here, n_s is the spectral tilt of the primordial power spectrum [Eq. (2.19)], whose current best estimation is $n_s = 0.963 \pm 0.012$ (68% CL) from WMAP7 (Komatsu et al., 2010). Therefore, any significant detection of f_{NL} above the value dictated by the consistency relation $f_{\text{NL}} \sim 0.017$ will rule out the single field inflationary models. The current limit from the WMAP 7-year data is $f_{\text{NL}} = 32 \pm 21$ (68% CL) (Komatsu et al., 2010), and from the SDSS is $f_{\text{NL}} = 31^{+16}_{-27}$ (68% CL) (Slosar et al., 2008). We shall further study the effect of local primordial non-Gaussianity on the galaxy bispectrum and the galaxy-galaxy, galaxy-CMB weak gravitational lensing in Chapter 5 and Chapter 6, respectively.

Local type primordial non-Gaussianity is defined by the Bardeen's potential $\Phi(\mathbf{x})$ [Eq. (2.14)] in real space (Salopek & Bond, 1990; Gangui et al., 1994; Verde et al., 2000; Komatsu & Spergel, 2001):

$$\Phi(\mathbf{x}) = \phi(\mathbf{x}) + f_{\text{NL}} [\phi^2(\mathbf{x}) - \langle \phi^2 \rangle] + \mathcal{O}(\phi^3), \quad (2.63)$$

where $\phi(\mathbf{x})$ is a Gaussian random field. The primordial bispectrum generated by the local

non-Gaussianity¹⁰ is

$$B_\Phi(k_1, k_2, k_3) = 2f_{\text{NL}} [P_\phi(k_1)P_\phi(k_2) + (2 \text{ cyclic})]. \quad (2.64)$$

We can find the linear matter bispectrum by linearly evolving the primordial bispectrum [Eq. (2.64)] to present using equation (2.18):

$$B_L(k_1, k_2, k_3; z) = 2f_{\text{NL}} D^3(z) \mathcal{M}(k_1) \mathcal{M}(k_2) \mathcal{M}(k_3) [P_\phi(k_1)P_\phi(k_2) + (2 \text{ cyclic})]. \quad (2.65)$$

Using the linear bispectrum above, we calculate the non-Gaussian correction to the nonlinear power spectrum as

$$\begin{aligned} \Delta P_{X, \text{nG}}(k, z) = & 4f_{\text{NL}} D^3(z) K_1^{(s)}(\mathbf{k}) \mathcal{M}(k) \int \frac{d^3 q}{(2\pi)^3} \mathcal{M}(q) \mathcal{M}(|\mathbf{k} - \mathbf{q}|) \\ & \times K_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) P_\phi(q) [2P_\phi(k) + P_\phi(|\mathbf{k} - \mathbf{q}|)]. \end{aligned} \quad (2.66)$$

Note that we ignore the nonlinearity in the linear power spectrum generated by equation (2.63), and use a linear approximation as

$$P_L(k, z) \simeq \mathcal{M}^2(k) D^2(z) P_\phi(k). \quad (2.67)$$

This approximation is valid up to slight rescaling of amplitude and slope of the primordial curvature power spectrum. For more discussion, see, Section II of McDonald (2008).

2.4 Nonlinear matter power spectrum in real space

For a nonlinear matter power spectrum in real space, we can simply use the perturbative solution for $\delta_{\mathbf{k}}(\tau)$ in equation (2.22). That is,

$$\begin{aligned} K_1^{(s)}(\mathbf{q}_1) &= 1 \\ K_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) &= F_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \\ K_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) &= F_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \end{aligned}$$

where $F_2^{(s)}$ and $F_3^{(s)}$ are presented in equation (2.32), and equation (2.34), respectively.

¹⁰For derivation, see Appendix C.

2.4.1 Gaussian case

By substituting this kernels and equation (2.58), we calculate the matter power spectrum in real space as

$$P_m(k, z) = D^2(z)P_L(k) + D^4(z)[P_{m,22}(k) + 2P_{m,13}(k)], \quad (2.68)$$

where $D(z)$ is the linear growth factor and

$$P_{m,22}(k) = \frac{1}{98} \frac{k^3}{(2\pi)^2} \int dr P_L(kr) \int_{-1}^1 dx P_L(k\sqrt{1+r^2-2rx}) \left[\frac{7x+3r-10rx^2}{1+r^2-2rx} \right]^2 \quad (2.69)$$

$$P_{m,13}(k) = \frac{1}{504} \frac{k^3}{(2\pi)^2} P_L(k) \int dr P_L(kr) \times \left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^3}(r^2-1)^3(7r^2+2) \ln \left(\frac{r+1}{|r-1|} \right) \right]. \quad (2.70)$$

Here, $P_L(k)$ is calculated at present where linear growth factor is normalized to be unity. This form is practical useful as, for given linear power spectrum, we only need to calculate the integration once for a give redshift. Then, the non-linear matter power spectrum for different redshifts can be calculated by simple rescaling of P_{11} , P_{22} and P_{13} with corresponding powers of linear growth factor.

We show that, in the quasi-nonlinear regime at high redshift, this *analytic* expression accurately models the nonlinear evolution of the matter power spectrum from a series of N-body simulations we run in Chapter 3. We also verify the result against the matter power spectrum from Millennium Simulation (Springel et al., 2005) in Section 4.2.

2.4.2 Non-Gaussianity case

We also calculate the leading order non-Gaussian term due to the local type primordial non-Gaussianity from equation (2.66).

$$\begin{aligned} \Delta P_{m,nG}(k, z) = & \frac{3}{7} f_{NL} H_0^2 \Omega_m D^3(z) \frac{k}{(2\pi)^2} \int \frac{dr}{r} \frac{P_L(kr)}{T(kr)} \int_{-1}^1 dx \left(\frac{7x+3r-10rx^2}{1+r^2-2rx} \right) \\ & \times \left[2P_L(k)(1+r^2-2rx) \frac{T(k\sqrt{1+r^2-2rx})}{T(k)} \right. \\ & \left. + \frac{P_L(k\sqrt{1+r^2-2rx})}{1+r^2-2rx} \frac{T(k)}{T(k\sqrt{1+r^2-2rx})} \right] \end{aligned} \quad (2.71)$$

This equation is first derived from Taruya et al. (2008), and they find that non-Gaussianity signal in matter power spectrum is so tiny that gigantic space based survey with survey volume of $100 [\text{Gpc}^3/h^3]$ only detect with large uncertainty ($\Delta f_{NL} \simeq 300$).

2.5 Nonlinear galaxy power spectrum in real space

In galaxy surveys, what we observe are galaxies, not a matter fluctuation. Since galaxies are the biased tracers of the underlying matter fluctuation, we have to understand how galaxy distribution and the matter fluctuation are related. This relation is known to be very complicated, because we have to understand the complex galaxy formation processes as well as the dark matter halo formation processes for given matter fluctuation in order to calculate the relation from the first principle. Both of which are the subject of the forefront research and need to be investigated further.

We simplify the situation by assuming that the galaxy formation and halo formation are local processes. This assumption is valid on large enough scale, which may include the *quasi nonlinear* scale where PT models the nonlinear evolution very well. Then, the galaxy over/under density at a given position depends only on the matter fluctuation at the same position. Therefore, the galaxy density contrast $\delta_g(\mathbf{x})$ can be Taylor-expanded with respect to the smoothed matter density contrast

$$\delta_R(\mathbf{x}) = \int d^3y \delta(\mathbf{y}) \mathcal{W}_R(\mathbf{x} - \mathbf{y})$$

as

$$\delta_g(\mathbf{x}) = \epsilon + c_1 \delta_R(\mathbf{x}) + \frac{c_2}{2} [\delta_R^2(\mathbf{x}) - \langle \delta_R^2 \rangle] + \frac{c_3}{6} \delta_R^3(\mathbf{x}) + \dots, \quad (2.72)$$

where $\langle \delta^2 \rangle$ is subtracted in order to ensure $\langle \delta_g \rangle = 0$ (McDonald, 2006). Here, $\mathcal{W}_R(r)$ is the smoothing (filtering) function, and $\bar{W}_R(k)$ is its Fourier transform¹¹. We also introduce the stochastic parameter ϵ which quantifies the “stochasticity” of galaxy bias, i.e. the relation between $\delta_g(\mathbf{x})$ and $\delta_R(\mathbf{x})$ is not completely deterministic, but contains some noise with zero mean, $\langle \epsilon \rangle = 0$ (e.g., Yoshikawa et al. (2001), and reference therein). We further assume that the stochasticity is a white noise, and is uncorrelated with the density fluctuations i.e., $\langle \epsilon^2(\mathbf{k}) \rangle \equiv \epsilon_0^2$, $\langle \epsilon \delta_R \rangle = 0$. The coefficients of expansion, c_n ’s, encode the detailed formation history of galaxies, and may vary for different morphological types, colors, flux limits, etc.

By using a convolution theorem, we calculate the Fourier transform of the local bias expansion of equation (2.72)

$$\begin{aligned} \delta_g(\mathbf{k}) = & \epsilon(\mathbf{k}) + c_1 \delta_R(\mathbf{k}) + \frac{c_2}{2} \int \frac{d^3q_1}{(2\pi)^3} \int d^3q_2 \delta_R(\mathbf{q}_1) \delta_R(\mathbf{q}_2) \delta_D(\mathbf{k} - \mathbf{q}_{12}) \\ & + \frac{c_3}{6} \int \frac{d^3q_1}{(2\pi)^3} \int \frac{d^3q_2}{(2\pi)^3} \int d^3q_3 \delta_R(\mathbf{q}_1) \delta_R(\mathbf{q}_2) \delta_R(\mathbf{q}_3) \delta_D(\mathbf{k} - \mathbf{q}_{123}), \end{aligned} \quad (2.73)$$

¹¹For the notational simplicity, we shall drop the tilde, but it should be clear from the argument whether the filtering function is defined in real space or Fourier space.

in terms of the smoothed non-linear density field $\delta_R(\mathbf{k})$:

$$\delta_R(\mathbf{k}) \equiv \mathcal{W}_R(k) \left[\delta^{(1)}(\mathbf{k}) + \delta^{(2)}(\mathbf{k}) + \delta^{(3)}(\mathbf{k}) + \dots \right]. \quad (2.74)$$

Here, $\delta^{(n)}(\mathbf{k})$ denotes the n -th order perturbation theory solution in equation (2.22). We find the kernel for the real space galaxy density contrast by substituting equation (2.74) into equation (2.73).

$$\begin{aligned} K_1^{(s)}(\mathbf{q}_1) &= c_1 \mathcal{W}_R(q_1) \\ K_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) &= \frac{c_2}{2} \mathcal{W}_R(q_1) \mathcal{W}_R(q_2) + c_1 F_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \mathcal{W}_R(q_{12}) \\ K_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) &= \frac{c_3}{6} \mathcal{W}_R(q_1) \mathcal{W}_R(q_2) \mathcal{W}_R(q_3) + c_1 F_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \mathcal{W}_R(q_{123}) \\ &\quad + \frac{c_2}{3} \left[F_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \mathcal{W}_R(q_3) \mathcal{W}_R(q_{12}) + (2 \text{ cyclic}) \right]. \end{aligned}$$

2.5.1 Gaussian case

As we assume that the stochastic parameter $\epsilon(\mathbf{k})$ is not correlated with the density field, we calculate the real space galaxy power spectrum in Gaussian case as

$$P_g(k, z) = \langle \epsilon^2 \rangle + D^2(z) P_{g,11}(k) + D^4(z) [P_{g,22}(k) + 2P_{g,13}(k)], \quad (2.75)$$

where

$$P_{g,11}(k) = c_1^2 \mathcal{W}_R^2(k) P_L(k) \quad (2.76)$$

is the linear bias term with linear matter power spectrum and $P_{g,22}$ and $P_{g,13}$ include the non-linear bias as well as the non-linear growth of the matter density field:

$$\begin{aligned} &P_{g,22}(k) \\ &= \frac{c_2^2}{2} \int \frac{d^3 q}{(2\pi)^3} \mathcal{W}_R^2(q) P_L(q) \mathcal{W}_R^2(|\mathbf{k} - \mathbf{q}|) P_L(|\mathbf{k} - \mathbf{q}|) \\ &\quad + 2c_1 c_2 \mathcal{W}_R(k) \int \frac{d^3 q}{(2\pi)^3} \mathcal{W}_R(q) P_L(q) \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) P_L(|\mathbf{k} - \mathbf{q}|) F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \\ &\quad + 2c_1^2 \mathcal{W}_R^2(k) \int \frac{d^3 q}{(2\pi)^3} P_L(q) P_L(|\mathbf{k} - \mathbf{q}|) \left[F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \right]^2 \end{aligned} \quad (2.77)$$

$$\begin{aligned} &P_{g,13}(k) \\ &= \frac{1}{2} c_1 c_3 \mathcal{W}_R^2(k) P_L(k) \sigma_R^2 + 3c_1^2 \mathcal{W}_R^2(k) P_L(k) \int \frac{d^3 q}{(2\pi)^3} P_L(q) F_3^{(s)}(\mathbf{q}, -\mathbf{q}, \mathbf{k}) \\ &\quad + 2c_1 c_2 \mathcal{W}_R(k) P_L(k) \int \frac{d^3 q}{(2\pi)^3} P_L(q) \mathcal{W}_R(q) \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) F_2(\mathbf{k}, -\mathbf{q}). \end{aligned} \quad (2.78)$$

Here,

$$\sigma_R^2 = \int \frac{d^3q}{(2\pi)^3} P_L(q) |\tilde{W}_R(q)|^2$$

is the root-mean-squared (r.m.s.) value of the smoothed linear density contrast at $z = 0$.

This equation is first derived in Smith et al. (2007) in the context of HaloPT, but they found the poor agreement between equation (2.75) and the halo power spectrum directly calculated from N-body simulation. However, it does not necessarily mean that the local bias ansatz of equation (2.72) is wrong. We rather attribute the failure of their modeling to the inaccurate modeling of the bias parameters (c_1 , c_2 and c_3) from the halo model. For example, the halo/galaxy power spectrum driven from local bias successfully models the halo/galaxy power spectrum from Millennium Simulation when fitting nonlinear bias parameters in Chapter 4.

Instead of using the bias parameters from the halo model, we shall treat the bias parameters as free parameters, and fit them to the observed galaxy power spectrum¹². In order to convert equation (2.75) into the practically useful form for fitting, we need to re-parametrize the bias parameters. It is because the theoretical template for fitting galaxy power spectrum shown in equation (2.75) has a few problems, as it was first pointed out by McDonald (2006). First, $P_{g,13}(k)$ in equation (2.75) contains σ_R^2 , which diverges, or is sensitive to the details of the small scale treatment, e.g. imposing a cut-off scale, choosing particular smoothing function, etc. Second, the first term in equation (2.77), one proportional to c_2^2 , approaches to a constant value on large scale limit i.e., $k \rightarrow 0$. The constant value can be large depending on the spectral index, or, again, sensitive to the small scale treatment.

In order to avoid these problems, we re-define the nonlinear bias parameters such that all terms sensitive to the small-scale treatment are absorbed into the parametrization. In other words, as we are interested in the power spectrum on sufficiently large scales, $k \ll 1/R$, we want to make the effect of small scale smoothing to be shown up only through the value of the bias parameters. On such large scales, we could approximate $W_R(k) = 1$, and the last term of $P_{g,13}(k)$ (Eq. [2.78]) is simply proportional to the linear power spectrum, and the proportionality constant depends only on the smoothing scale R . That is, if we

¹²For the goodness of the fitting method including the effect of fitting to extracting the cosmological parameters, see Chapter 4.

rewrite

$$\begin{aligned}
& 2c_1c_2\mathcal{W}_R(k)P_L(k)\int\frac{d^3q}{(2\pi)^3}P_L(q)\mathcal{W}_R(q)\mathcal{W}_R(|\mathbf{k}-\mathbf{q}|)F_2(\mathbf{k},-\mathbf{q}) \\
& \equiv 4c_1c_2\sigma_R^2\mathcal{G}_R(k)\mathcal{W}_R^2(k)P_L(k),
\end{aligned} \tag{2.79}$$

then, $\mathcal{G}_R(k)$ approaches to the constant value

$$\mathcal{G}_R(k) \rightarrow \frac{13}{84} + \frac{1}{4\sigma_R^2} \int \frac{d^3q}{(2\pi)^3} P_L(q) \mathcal{W}_R(q) \frac{\sin(qR)}{qR}, \tag{2.80}$$

as $k \rightarrow 0$ ¹³. See Figure 5.7 for the shape of $\mathcal{G}_R(k)$ for $R = 1, 2, 5$ and $10 \text{ Mpc}/h$. We also show the large scale asymptotic value of $\mathcal{G}_R(0)$ as a function of R in Figure 5.8. With the definition of $\mathcal{G}_R(k)$ in equation (2.79) and the non-linear matter power spectrum in equation (2.68), we rewrite the equation (2.75) as

$$\begin{aligned}
P_g(k, z) = & \langle \epsilon^2 \rangle + c_1^2 \mathcal{W}_R^2(k) P_m(k, z) \\
& + D^2(z) [c_1c_3\sigma_R^2 + 8c_1c_2\sigma_R^2\mathcal{G}_R(k)] \mathcal{W}_R^2(k) P_L(k, z) \\
& + \frac{c_2^2}{2} \int \frac{d^3q}{(2\pi)^3} \mathcal{W}_R^2(q) P_L(q, z) \mathcal{W}_R^2(|\mathbf{k}-\mathbf{q}|) P_L(|\mathbf{k}-\mathbf{q}|, z) \\
& + 2c_1c_2\mathcal{W}_R(k) \int \frac{d^3q}{(2\pi)^3} \mathcal{W}_R(q) P_L(q, z) \\
& \times \mathcal{W}_R(|\mathbf{k}-\mathbf{q}|) P_L(|\mathbf{k}-\mathbf{q}|, z) F_2^{(s)}(\mathbf{q}, \mathbf{k}-\mathbf{q}).
\end{aligned} \tag{2.82}$$

We re-parametrize the nonlinear bias parameters as,

$$P_0 = \langle \epsilon^2 \rangle + D^4(z) \frac{c_2^2}{2} \int \frac{dq}{2\pi^2} q^2 [P_L(q) \mathcal{W}_R^2(q)]^2 \tag{2.83}$$

$$b_1^2 = c_1^2 + D^2(z) [c_1c_3\sigma_R^2 + 8c_1c_2\mathcal{G}_R(k)\sigma_R^2] \tag{2.84}$$

$$b_2 = \frac{c_2}{b_1}, \tag{2.85}$$

then, the galaxy power spectrum becomes

$$P_g(k, z) = P_0 + b_1^2 [\mathcal{W}_R^2(k) P_m(k, z) + b_2 D^4(z) P_{b2}(k) + b_2^2 D^4(z) P_{b22}(k)], \tag{2.86}$$

¹³For general window function $\mathcal{W}_R(k)$, as $k \rightarrow 0$,

$$\begin{aligned}
& \int \frac{d^3q}{(2\pi)^3} P_L(q) \mathcal{W}_R(q) \mathcal{W}_R(|\mathbf{k}-\mathbf{q}|) F_2^{(s)}(\mathbf{k}, -\mathbf{q}) \\
& \rightarrow \frac{17}{21} \sigma_R^2 + \frac{1}{6} \int \frac{d^3q}{(2\pi)^3} q P_L(q) \mathcal{W}_R(q) \frac{d\mathcal{W}_R(q)}{dq}.
\end{aligned} \tag{2.81}$$

Therefore, if we do not employ the smoothing function, i.e. $\mathcal{W}_R(k) = 1$, the integration becomes $17/21\sigma_R^2$, and hence, the last term of $P_{g,13}(k)$ is simply $34/21c_1c_2\sigma^2P_L(k)$. This result coincides with McDonald (2006).

where $P_{b2}(k)$ and $P_{b22}(k)$ are given by

$$P_{b2}(k) = 2\mathcal{W}_R(k) \int \frac{d^3q}{(2\pi)^3} \mathcal{W}_R(q) P_L(q) \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) P_L(|\mathbf{k} - \mathbf{q}|) F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \quad (2.87)$$

$$P_{b22}(k) = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \mathcal{W}_R^2(q) P_L(q) [\mathcal{W}_R^2(|\mathbf{k} - \mathbf{q}|) P_L(|\mathbf{k} - \mathbf{q}|) - \mathcal{W}_R^2(q) P_L(q)]. \quad (2.88)$$

Note that equation (2.86) is the same as the original equation up to next-to-leading order, e.g. $\sigma_R^2 P_m \simeq \sigma_R^2 P_L + \mathcal{O}(P_L^3)$. As we have desired, the terms depending on the smoothing scale R are absorbed into the newly defined bias parameters P_0 , b_1 and b_2 , and $P_m(k)$, $P_{b2}(k)$, $P_{b22}(k)$ are independent of the smoothing scale on large scales, $k \gg 1/R$.

Note that b_1 we defined here reduces to the ‘effective bias’ of Heavens et al. (1998) in the $R \rightarrow 0$ limit, and in $k \rightarrow 0$ limit, equation (2.86) approaches to

$$P_g(k) \rightarrow P_0 + b_1^2 P_m(k),$$

the usual linear bias model plus a constant.

The ‘re-parametrized’ nonlinear bias parameters, P_0 , b_1 , b_2 , encode the complex galaxy formation processes, which will be very hard to model from the first principle (Smith et al., 2007). Nevertheless, the nonlinear galaxy power spectrum we calculate here has to be the ‘right’ prescription as long as the locality of bias assumption is correct in the quasi-nonlinear regime. In Chapter 4, we tested the nonlinear bias model in equation (2.86) against the halos/galaxies power spectrum of the Millennium Simulation (Springel et al., 2005). In order to test the prescription itself, we set P_0 , b_1 and b_2 as free parameters, and fit the measured power spectrum from Millennium simulation with equation (2.86). We found that nonlinear bias model provides a significantly better fitting than the linear bias model. In addition to that, we could reproduce the *correct* distance scales within a statistical error-bar, when marginalizing over three *free* nonlinear bias parameters.

2.5.2 Galaxy-matter cross power spectrum

As we shall marginalize over the bias parameters, the more do we add information about bias parameters, the better can we estimate the other cosmological parameters. The galaxy-matter cross power spectrum at high redshift can be a source of such an additional information, as it is proportional to the galaxy density contrast; thus, it also depends on the bias parameter. We can measure the galaxy-matter cross power spectrum from the galaxy-galaxy, and galaxy-CMB weak lensing measurements¹⁴.

¹⁴We study the galaxy-galaxy, and galaxy-CMB weak lensing on large scales in Chapter 6.

The fastest way of calculating the galaxy-matter cross power spectrum is using the calculation of galaxy-galaxy power spectrum. Let us abbreviate equation (2.73) as

$$\delta_g(\mathbf{k}) = \epsilon(\mathbf{k}) + c_1 \delta_R(\mathbf{k}) + c_2 \delta_g^{(2)}(\mathbf{k}) + c_3 \delta_g^{(3)}(\mathbf{k}). \quad (2.89)$$

Then, we can think of calculating the galaxy-galaxy power spectrum as

$$\begin{aligned} & \langle \delta_g(\mathbf{k}) \delta_g(\mathbf{k}') \rangle \\ &= \langle \epsilon^2 \rangle + c_1 \langle \delta_R(\mathbf{k}) \delta_g(\mathbf{k}') \rangle + \langle (c_2 \delta_g^{(2)}(\mathbf{k}) + c_3 \delta_g^{(3)}(\mathbf{k})) \delta_g(\mathbf{k}') \rangle \\ &= \langle \epsilon^2 \rangle + c_1 \langle \delta_R(\mathbf{k}) \delta_g(\mathbf{k}') \rangle + c_1 \langle [c_2 \delta_g^{(2)}(\mathbf{k}) + c_3 \delta_g^{(3)}(\mathbf{k})] \delta_R(\mathbf{k}') \rangle + \dots \\ &= \langle \epsilon^2 \rangle + c_1 [\langle \delta_R(\mathbf{k}) \delta_g(\mathbf{k}') \rangle + \langle \delta_R(\mathbf{k}') \delta_g(\mathbf{k}) \rangle - c_1 \langle \delta_R(\mathbf{k}) \delta_R(\mathbf{k}') \rangle] + \dots \end{aligned} \quad (2.90)$$

From equation (2.90), it is clear that adding up the terms proportional to c_1 in $P_g(\mathbf{k})$ are the same as $c_1 [2P_{gm}(\mathbf{k}) - c_1 P_m(k)]$. Therefore, the nonlinear galaxy-matter cross correlation function is

$$\begin{aligned} P_{gm}(k, z) &= c_1 \mathcal{W}_R^2(k) P_m(k, z) + D^2(z) \left[\frac{c_3}{2} \sigma_R^2 + 4c_2 \sigma_R^2 \mathcal{G}_R(k) \right] \mathcal{W}_R^2(k) P_L(k, z) \\ &+ c_2 \mathcal{W}_R(k) \int \frac{d^3 q}{(2\pi)^3} \mathcal{W}_R(q) P_L(q, z) \\ &\times \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) P_L(|\mathbf{k} - \mathbf{q}|, z) F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}). \end{aligned} \quad (2.91)$$

We also re-parametrize the bias for this case,

$$\bar{b}_1 = c_1 + D^2(z) \left[\frac{c_3}{2} \sigma_R^2 + 4c_2 \sigma_R^2 \mathcal{G}_R(k) \right] \quad (2.92)$$

$$\bar{b}_2 = \frac{c_2}{\bar{b}_1}, \quad (2.93)$$

so that the galaxy-matter cross power becomes

$$P_{gm}(k, z) = \bar{b}_1 \left[\mathcal{W}_R^2(k) P_m(k, z) + \frac{\bar{b}_2}{2} D^4(z) P_{b2}(k) \right], \quad (2.94)$$

where $P_{b2}(k)$ is defined in equation (2.87). Note that when $\sigma_R \ll 1$, $\bar{b}_1 \sim b_1$ and $\bar{b}_2 \simeq b_2$.

2.5.3 non-Gaussian case

What about the non-Gaussian correction term? We calculate the non-Gaussian term by substituting the real space galaxy kernels into equation (2.66).

$$\begin{aligned}
& \Delta P_{g,nG}(k, z) \\
&= 2c_1 f_{\text{NL}} D^3(z) \mathcal{W}_R(k) \mathcal{M}(k) \int \frac{d^3 q}{(2\pi)^3} \mathcal{M}(q) \mathcal{M}(|\mathbf{k} - \mathbf{q}|) P_\phi(q) [2P_\phi(k) + P_\phi(|\mathbf{k} - \mathbf{q}|)] \\
&\quad \times \left[c_2 \mathcal{W}_R(q) \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) + 2c_1 \mathcal{W}_R(k) F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \right] \\
&= c_1^2 \mathcal{W}_R^2(k) \Delta P_{m,ng}(k, z) + 4c_1 c_2 D^3(z) \sigma_R^2 f_{\text{NL}} \mathcal{W}_R(k) \mathcal{F}_R(k) \frac{P_L(k)}{M(k)} \tag{2.95}
\end{aligned}$$

Here, $\Delta P_{m,nG}$ is the non-Gaussian correction to the matter power spectrum, and

$$\mathcal{F}_R(k) \equiv \frac{1}{2\sigma_R^2} \int \frac{d^3 q}{(2\pi)^3} \mathcal{M}_R(q) \mathcal{M}_R(|\mathbf{k} - \mathbf{q}|) P_\phi(q) \left[\frac{P_\phi(|\mathbf{k} - \mathbf{q}|)}{P_\phi(k)} + 2 \right] \tag{2.96}$$

is a function which is unity on large scales ($k \ll 1/R$, See, e. g. Matarrese & Verde, 2008). See Figure 5.6 for the shape of $\mathcal{F}_R(k)$ for $R = 1, 2, 5$ and $10 \text{ Mpc}/h$.

The first term in equation (2.95) is simply the non-Gaussian matter power spectrum multiplied by the linear bias factor. The second term in Eq. (2.95) is the non-Gaussianity term generated by non-linear bias, and shows the same behavior as the scale dependent bias from local type primordial non-Gaussianity¹⁵. In fact, this term reduces to the result of MLB formula (Matarrese et al., 1986; Matarrese & Verde, 2008, see, Appendix K.2 for derivation)

$$\Delta P_{g,nG}(k, z) = 4c_1(c_1 - 1)\delta_c D(z) \alpha f_{\text{NL}} \mathcal{W}_R(k) \mathcal{F}_R(k) \frac{P_L(k)}{M(k)} \tag{2.97}$$

in the linear regime and for the high-peak limit of the halo model¹⁶. It is sufficient to show that $c_1 c_2 D^2(z) \sigma_R^2$ becomes $\alpha c_1(c_1 - 1)\delta_c$ in the high-peak limit. Consider the bias parameters from the halo model: (Scoccimarro et al., 2001b)

$$c_1 = 1 + \frac{\alpha\nu^2 - 1}{\delta_c} + \frac{2p/\delta_c}{1 + (\alpha\nu^2)^p} \tag{2.98}$$

$$c_2 = \frac{8}{21}(c_1 - 1) + \frac{\alpha\nu^2}{\delta_c^2} (\alpha\nu^2 - 3) + \frac{2p/\delta_c}{1 + (\alpha\nu^2)^p} \left(\frac{1 + 2p}{\delta_c} + 2 \frac{\alpha\nu^2 - 1}{\delta_c} \right), \tag{2.99}$$

where $\delta_c \simeq 1.686$ is the critical overdensity above which halo forms, and $\nu \equiv \delta_c/(D^2(z)\sigma_R^2)$. For Press-Schechter mass function (spherical collapse, Press & Schechter, 1974; Mo & White,

¹⁵For the scale dependent bias, see the introduction in Chapter 5, and Appendix I.3.

¹⁶ α here is the same as q in Carbone et al. (2008). We reserve q for the Fourier space measure.

1996), $\alpha = 1$, $p = 0$, and for Sheth-Tormen mass function (ellipsoidal collapse, Sheth & Tormen, 1999; Sheth et al., 2001), $\alpha = 0.75$, $p = 0.3$. And in the high-peak limit ($\nu \gg 1$), we approximate $c_1 - 1 \simeq \alpha\nu^2/\delta_c$, and $c_2 \simeq \alpha^2\nu^4/\delta_c^2$. Therefore,

$$c_1 c_2 D^2(z) \sigma_R^2 \simeq c_1 \alpha^2 \nu^4 \left(\frac{D^2(z) \sigma_R^2}{\delta_c^2} \right) = c_1 \alpha^2 \nu^2 \simeq \alpha c_1 (c_1 - 1) \delta_c. \quad (2.100)$$

This relation motivate us to define a new bias parameter $\tilde{b}_2 \equiv \sigma_R^2 D^2(z) c_2 / c_1$, which approaches $\tilde{b}_2 \rightarrow \alpha \delta_c$ for the high peak limit of the halo model.

By using a re-parametrized bias, b_1 and $\tilde{b}_2$, the non-Gaussian correction term becomes

$$\begin{aligned} & \Delta P_{g,nG}(k, z) \\ &= b_1^2 \left[\mathcal{W}_R^2(k) \Delta P_{m,ng}(k, z) + 6\tilde{b}_2 f_{\text{NL}} D(z) \mathcal{W}_R(k) \mathcal{F}_R(k) \frac{H_0^2 \Omega_m P_L(k)}{k^2 T(k)} \right], \end{aligned} \quad (2.101)$$

and on large scales ($k \ll 1/R$), for high-peak, the formula reduces to the usual form in the literature (Dalal et al., 2008; Matarrese & Verde, 2008; Slosar et al., 2008; Afshordi & Tolley, 2008; Taruya et al., 2008; McDonald, 2008; Sefusatti, 2009):

$$\Delta P_{g,nG}(k, z) = b_1^2 \left[\Delta P_{m,ng}(k, z) + 6\alpha \delta_c f_{\text{NL}} D(z) \frac{H_0^2 \Omega_m P_L(k)}{k^2 T(k)} \right]. \quad (2.102)$$

Although equation (2.101) coincides with equation (2.102) for high peak limit, it may not be the dominant contribution of scale-dependent bias for intermediate size peaks where the nonlinear bias b_2 is actually small. Recent study based on Peak Background Split method (Giannantonio & Porciani, 2010) suggests that for non-Gaussian case, the local ansatz [Eq. (2.72)] has to be modified to include the effect of Gaussian piece of gravitational potential $\phi(\mathbf{x})$ directly as

$$\delta_g(\mathbf{x}) = b_{10} \delta(\mathbf{x}) + b_{01} \phi(\mathbf{x}) + \frac{1}{2!} (b_{20} \delta^2(\mathbf{x}) + 2b_{11} \delta(\mathbf{x}) \phi(\mathbf{x}) + b_{02} \phi^2(\mathbf{x})) + \dots, \quad (2.103)$$

where b_{ij} s are bias parameters. If this holds, the non-Gaussianity signal from the power spectrum of very massive clusters (where $\tilde{b}_2$ is indeed close to $\alpha \delta_c$) is expected to be twice as high as the scale dependent bias in equation (2.102).

2.6 Nonlinear mater power spectrum in redshift space

In the previous sections, we have calculated the matter power spectrum and the galaxy power spectrum in *real* space. By real space, we mean an idealistic universe where

we can observe the *true* distance of the galaxies (or matter particles) relative to us. With galaxy survey alone, however, we cannot measure the true distance to the galaxies, as we infer the distance to a galaxy from the galaxy's spectral line shift by assuming the Hubble law. The problem here is that the observed spectral shift depends not only on the position of the galaxy (as a result of the expansion of the Universe), but also on the peculiar velocity of the galaxy. As a result, the three-dimensional map of the galaxies generated from galaxy surveys is different from the real space galaxy distribution. In contrast to the real space, we call the *observed* coordinate of galaxies the *redshift space*, and the radial distortion in the redshift space due to the peculiar velocity is called *redshift space distortion*.

We formulate the redshift space position vector $\mathbf{s}$ as follow:

$$\mathbf{s} = \mathbf{x} + (1+z) \frac{v_r(\mathbf{x})}{H(z)} \hat{\mathbf{r}}. \quad (2.104)$$

Here, $\mathbf{x}$ denotes the real space comoving position vector, and z denotes the redshift of galaxy without peculiar velocity, $H(z)$ is the Hubble parameter at that redshift, and v_r denotes the line-of-sight directional peculiar velocity. As redshift space distortion is due to the peculiar velocity, we can model it by using the peculiar velocity solution $\theta_{\mathbf{k}}(\tau)$ (Eq. [2.23]) of perturbation theory. In this section and the next section, we calculate the matter power spectrum and the galaxy power spectrum in redshift space, respectively.

How does the real space power spectrum changed under the redshift space distortion? In order to simplify the analysis, we make the *plane parallel* approximation that the galaxies are so far away that the radial direction is parallel to the $\hat{\mathbf{z}}$ direction¹⁷. Also, we define the *reduced* velocity field $\mathbf{u} \equiv \mathbf{v}/(f\mathcal{H})$ so that equation (2.104) becomes

$$\mathbf{s} = \mathbf{x} + f u_z(\mathbf{x}) \hat{\mathbf{z}}. \quad (2.105)$$

As $\mathbf{u}(\mathbf{k}) \equiv \mathbf{v}(\mathbf{k})/(f\mathcal{H}) = -i\mathbf{k}\theta_{\mathbf{k}}(\tau)/(k^2 f\mathcal{H})$, the Fourier transform of $u_z(\mathbf{x})$ becomes

$$\begin{aligned} u_z(\mathbf{k}, \tau) &= \frac{i\mu}{k} \sum_{n=1}^{\infty} \int \frac{d^3 q_1}{(2\pi)^3} \cdots \int \frac{d^3 q_n - 1}{(2\pi)^3} \int d^3 q_n \delta^D(\mathbf{k} - \sum_{i=1}^n \mathbf{q}_i) \\ &\quad \times G_n^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n) \delta_1(\mathbf{q}_1, \tau) \cdots \delta_1(\mathbf{q}_n, \tau) \equiv \frac{i\mu}{k} \eta(\mathbf{k}, \tau), \end{aligned} \quad (2.106)$$

where $\mu = \mathbf{k} \cdot \hat{\mathbf{z}}/k$ is the directional cosine between the wavenumber vector $\hat{\mathbf{k}}$ and the line of sight direction $\hat{\mathbf{z}}$. Note that the time evolution of the new variable $u_z(\mathbf{k})$ only comes from the linear density contrast.

¹⁷For a redshift space distortion including a light-cone effect, see, e.g. de Lai & Starkman (1998); Yamamoto et al. (1999); Nishioka & Yamamoto (2000); Wagner et al. (2008).

Let us denote the real space over-density as $\delta_r(\mathbf{x})$, and the redshift space over-density as $\delta_s(\mathbf{s})$. The mass conservation relates the measure in real space d^3r and that in redshift space d^3s as

$$(1 + \delta_s(\mathbf{s}))d^3s = (1 + \delta_r(\mathbf{x}))d^3x. \quad (2.107)$$

By using this relation, we find the *exact* relation between two over-densities in Fourier space (Scoccimarro, 2004; Matsubara, 2008).

$$\begin{aligned} \delta_s(\mathbf{k}) &= \int d^3s [1 + \delta_s(\mathbf{s})] e^{-i\mathbf{k}\cdot\mathbf{s}} - \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \int d^3x [1 + \delta_r(\mathbf{x})] e^{-i\mathbf{k}\cdot[\mathbf{x} + f u_z(\mathbf{x})\hat{\mathbf{z}}]} - \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \delta_r(\mathbf{k}) + \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \left(e^{-ik_z f u_z(\mathbf{x})} - 1 \right) [1 + \delta_r(\mathbf{x})] \end{aligned} \quad (2.108)$$

In order to calculate the 3rd order power spectrum, we Taylor-expand the exponential function up to 3rd order:

$$\begin{aligned} \delta_s(\mathbf{k}) &= \delta_r(\mathbf{k}) + f\mu^2\eta(\mathbf{k}) - \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &\quad \times \left[ik_z f u_z(\mathbf{x})\delta_r(\mathbf{x}) + \frac{1}{2}k_z^2 f^2 u_z^2(\mathbf{x}) + \frac{1}{2}k_z^2 f^2 u_z^2(\mathbf{x})\delta_r(\mathbf{x}) - \frac{i}{6}k_z^3 f^3 u_z^3(\mathbf{x}) \right]. \end{aligned} \quad (2.109)$$

We calculate the 3rd order nonlinear matter kernels in redshift space from equation (2.109) and using the convolution theorem:

$$K_1^{(s)}(\mathbf{k}) = 1 + f\mu^2 \quad (2.110)$$

$$\begin{aligned} K_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) &= F_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) + f\mu^2 G_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \\ &\quad + \frac{fk\mu}{2} \left(\frac{q_{1z}}{q_1^2} + \frac{q_{2z}}{q_2^2} \right) + \frac{(fk\mu)^2}{2} \frac{q_{1z}q_{2z}}{q_1^2 q_2^2} \end{aligned} \quad (2.111)$$

$$\begin{aligned} K_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) &= F_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) + f\mu^2 G_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \\ &\quad + \frac{(fk\mu)^2}{6} \left(\frac{q_{1z}q_{2z}}{q_1^2 q_2^2} + \frac{q_{2z}q_{3z}}{q_2^2 q_3^2} + \frac{q_{3z}q_{1z}}{q_3^2 q_1^2} \right) + \frac{(fk\mu)^3}{6} \frac{q_{1z}q_{2z}q_{3z}}{q_1^2 q_2^2 q_3^2} \\ &\quad + \frac{fk\mu}{3} \left\{ F_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \frac{q_{3z}}{q_3^2} + (2 \text{ cyclic}) \right\} \\ &\quad + \frac{(fk\mu)^2}{3} \left\{ G_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \frac{q_{3z}q_{(1+2)z}}{q_3^2 |\mathbf{q}_1 + \mathbf{q}_2|^2} + (2 \text{ cyclic}) \right\} \\ &\quad + \frac{fk\mu}{3} \left\{ G_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \frac{q_{(1+2)z}}{|\mathbf{q}_1 + \mathbf{q}_2|^2} + (2 \text{ cyclic}) \right\}. \end{aligned} \quad (2.112)$$

Note that the kernels we present here coincide those in equation (13) of Heavens et al. (1998) when setting $b_1 = 1$ and $b_2 = 0$.

Before calculating the power spectrum, it is instructive to compare the result here with other formulas in the literature. In linear regime, equation (2.109) reduces to

$$\delta_s(\mathbf{k}) = (1 + f\mu^2)\delta^{(1)}(\mathbf{k}), \quad (2.113)$$

and the redshift space matter power spectrum becomes

$$P_s(k, \mu, z) = (1 + f\mu^2)^2 D^2(z) P_L(k), \quad (2.114)$$

as shown in Kaiser (1987). That is, as a result of non-linear mapping between real and redshift space, the redshift space matter power spectrum is no longer spherically symmetric. This is called ‘Kaiser effect’ in the literature. Note that the redshift space distortion effect is larger for the line of sight direction ($\mu = 1$), and it does not change the power spectrum along the direction perpendicular to the line of sight ($\mu = 0$).

If we pick up the linear terms in equation (2.109)

$$\delta_s(\mathbf{k}) = \delta_r(\mathbf{k}) + f\mu^2\eta(\mathbf{k}), \quad (2.115)$$

and use the third order solution of $\delta_r(\mathbf{k})$ and $\eta(\mathbf{k})$, the redshift space matter power spectrum reduces to the formula given in Scoccimarro (2004):

$$P_s(k, \mu, z) = P_{\delta\delta}(k, z) + 2f\mu^2 P_{\delta\theta}(k, z) + f^2\mu^4 P_{\theta\theta}(k, z). \quad (2.116)$$

Here, $P_{\delta\delta}(k, z)$ is the same as the non-linear matter power spectrum $P_m(k, z)$ in real space [Eq. (2.68)], $P_{\delta\theta}(k, z)$ is the non-linear density-velocity cross power spectrum

$$P_{\delta\theta}(k, z) = D^2(z)P_L(k) + D^4(z)[P_{\delta\theta,22}(k) + P_{\delta\delta,13}(k) + P_{\theta\theta,13}(k)], \quad (2.117)$$

and $P_{\theta\theta}(k, z)$ is the non-linear velocity power spectrum

$$P_{\theta\theta}(k, z) = D^2(z)P_L(k) + D^4(z)[P_{\theta\theta,22}(k) + 2P_{\theta\theta,13}(k)], \quad (2.118)$$

where $P_{\delta\delta,13}(k)$ is shown in equation (2.70), and

$$\begin{aligned}
P_{\delta\theta,22}(k) &= 2 \int \frac{d^3q}{(2\pi)^3} P_L(q) P_L(|\mathbf{k} - \mathbf{q}|) F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) G_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}), \\
&= \frac{1}{98} \frac{k^3}{(2\pi)^2} \int dr P_L(kr) \int_{-1}^1 dx P_L(k\sqrt{1+r^2-2rx}) \\
&\quad \times \frac{(7x+3r-10rx^2)(7x-r-6rx^2)}{(1+r^2-2rx)^2}, \tag{2.119}
\end{aligned}$$

$$\begin{aligned}
P_{\theta\theta,22}(k) &= 2 \int \frac{d^3q}{(2\pi)^3} P_L(q) P_L(|\mathbf{k} - \mathbf{q}|) \left[G_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \right]^2 \\
&= \frac{1}{98} \frac{k^3}{(2\pi)^2} \int dr P_L(kr) \int_{-1}^1 dx P_L(k\sqrt{1+r^2-2rx}) \left[\frac{7x-r-6rx^2}{1+r^2-2rx} \right]^2, \tag{2.120}
\end{aligned}$$

$$\begin{aligned}
P_{\theta\theta,13}(k) &= 3 P_L(k) \int \frac{d^3q}{(2\pi)^3} P_L(q) G_3^{(s)}(\mathbf{q}, \mathbf{k}, -\mathbf{q}) \\
&= \frac{1}{504} \frac{k^3}{(2\pi)^2} P_L(k) \int dr P_L(kr) \\
&\quad \times \left[\frac{36}{r^2} - 246 + 12r^2 - 18r^4 + \frac{9}{r^3} (r^2-1)^3 (r^2+2) \ln \left(\frac{1+r}{|1-r|} \right) \right]. \tag{2.121}
\end{aligned}$$

2.6.1 Gaussian case

Substituting these kernels into equation (2.58), we find the 3rd order nonlinear redshift space matter power spectrum. The integration becomes particularly easier when we align $\hat{\mathbf{q}}_z$ (or $\hat{\phi}_q$ in spherical polar coordinate) parallel to the direction of the wavenumber vector, $\hat{\mathbf{k}}$. With this coordinate choice, the line of sight component of $\mathbf{q}$ can be written as

$$q_z = q(\mu x - \sqrt{1-\mu^2} \sqrt{1-x^2} \cos \phi_q),$$

where x denotes the directional cosine between $\mathbf{q}$ and $\mathbf{k}$, and we can integrate some part of angular integration analytically (Matsubara, 2008). After the analytical angular integration, the 3rd order matter power spectrum in redshift space becomes

$$P_s(k, \mu, z) = D^2(z) P_{s,11}(k, \mu) + D^4(z) [P_{s,22}(k, \mu) + 2P_{s,13}(k, \mu)], \tag{2.122}$$

where

$$P_{s,11}(k, \mu) = (1 + f\mu^2)^2 P_L(k), \quad (2.123)$$

$$P_{s,22}(k, \mu) = \sum_{n,m} \mu^{2n} f^m \frac{k^3}{(2\pi)^2} \int_0^\infty dr P_L(kr) \int_{-1}^1 dx \\ \times P_L\left(k(1+r^2-2rx)^{1/2}\right) \frac{A_{nm}(r, x)}{(1+r^2-2rx)^2}, \quad (2.124)$$

$$P_{s,13}(k, \mu) = (1 + f\mu^2) P_L(k) \sum_{n,m} \mu^{2n} f^m \frac{k^3}{(2\pi)^2} \int dr P_L(kr) B_{nm}(r), \quad (2.125)$$

with

$$A_{00} = \frac{1}{98}(3r + 7x - 10rx^2)^2 \\ A_{11} = \frac{2}{49}(3r + 7x - 10rx^2)^2 \\ A_{12} = \frac{1}{28}(1 - x^2)(7 - 6r^2 - 42rx + 48r^2x^2) \\ A_{22} = \frac{1}{196}(-49 + 114r^2 + 714rx + 637x^2 - 942r^2x^2 - 1890rx^3 + 1416r^2x^4) \\ A_{23} = \frac{1}{14}(1 - x^2)(7 - 6r^2 - 42rx + 48r^2x^2) \\ A_{24} = \frac{3}{16}r^2(x - 1)^2(x + 1)^2 \\ A_{33} = \frac{1}{14}(-7 + 6r^2 + 54rx + 35x^2 - 66r^2x^2 - 110rx^3 + 88r^2x^4) \\ A_{34} = \frac{1}{8}(1 - x^2)(2 - 3r^2 - 12rx + 15r^2x^2) \\ A_{44} = \frac{1}{16}(-4 + 3r^2 + 24rx + 12x^2 - 30r^2x^2 - 40rx^3 + 35r^2x^4)$$

and

$$B_{00} = \frac{1}{504} \left[\frac{2}{r^2}(6 - 79r^2 + 50r^4 - 21r^6) + \frac{3}{r^3}(r^2 - 1)^3(2 + 7r^2) \ln \left(\frac{1+r}{|1-r|} \right) \right] \\ B_{11} = 3B_{00} \\ B_{12} = \frac{1}{336} \left[\frac{2}{r^2}(9 - 89r^2 - 33r^4 + 9r^6) - \frac{9}{r^3}(r^2 - 1)^4 \ln \left(\frac{1+r}{|1-r|} \right) \right] \\ B_{22} = \frac{1}{336} \left[\frac{2}{r^2}(9 - 109r^2 + 63r^4 - 27r^6) + \frac{9}{r^3}(r^2 - 1)^3(1 + 3r^2) \ln \left(\frac{1+r}{|1-r|} \right) \right] \\ B_{23} = -\frac{1}{3}.$$

This non-linear redshift space power spectrum shows more complicated angular dependence than Kaiser effect.

In Section 3.3, we compare the redshift power spectrum in equation (2.122) with redshift space matter power spectrum we measured from a series of N-body simulations.

2.6.2 non-Gaussian case

We calculate the non-Gaussian correction to the matter power spectrum in redshift space by substituting the kernels in equation (2.110) and equation (2.111) into equation (2.66):

$$\begin{aligned}
& \Delta P_s(k, \mu, z) \\
&= 4f_{\text{NL}} D^3(z) (1 + f\mu^2) \mathcal{M}(k) \int \frac{d^3 q}{(2\pi)^3} \mathcal{M}(q) \mathcal{M}(|\mathbf{k} - \mathbf{q}|) \\
&\quad \times P_\phi(q) [2P_\phi(k) + P_\phi(|\mathbf{k} - \mathbf{q}|)] \left[F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) + f\mu^2 G_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \right. \\
&\quad \left. + \frac{fk\mu}{2} \left\{ \frac{q_z}{q^2} + \frac{k\mu - q_z}{|\mathbf{k} - \mathbf{q}|^2} \right\} + \frac{(fk\mu)^2}{2} \frac{q_z(k\mu - q_z)}{q^2 |\mathbf{k} - \mathbf{q}|^2} \right]. \tag{2.126}
\end{aligned}$$

As is the case for the non-Gaussianity correction to the matter power spectrum in real space, equation (2.126) is parametrically smaller than the non-linear terms: $P_{s,22}(k, \mu)$ and $P_{s,13}(k, \mu)$.

2.7 Nonlinear galaxy power spectrum in redshift space

Finally, in this section, we combine all three non-linear effects on the galaxy power spectrum we shall measure from galaxy surveys: the galaxy power spectrum in redshift space. As real to redshift mapping is the same for the total matter and the galaxies, the redshift space density contrast is also given by equation (2.109), but changing the real space density contrast $\delta_r(\mathbf{k})$ to the real space galaxy density contrast $\delta_g(\mathbf{k})$:

$$\begin{aligned}
\delta_{gs}(\mathbf{k}) &= \delta_g(\mathbf{k}) + f\mu^2 \eta(\mathbf{k}) - \int d^3 x e^{-i\mathbf{k}\cdot\mathbf{x}} \\
&\quad \times \left[ik_z f u_z(\mathbf{x}) \delta_g(\mathbf{x}) + \frac{1}{2} k_z^2 f^2 u_z^2(\mathbf{x}) + \frac{1}{2} k_z^2 f^2 u_z^2(\mathbf{x}) \delta_g(\mathbf{x}) - \frac{i}{6} k_z^3 f^3 u_z^3(\mathbf{x}) \right], \tag{2.127}
\end{aligned}$$

again, $\mu \equiv k_z/k$ is the cosine between wave vector $\mathbf{k}$ and the line of sight direction, and $\eta(\mathbf{k})$ is defined as $\tilde{u}_z(\mathbf{k}) \equiv i\mu\eta(\mathbf{k})/k$ (see, equation (2.106)). Note that the first two terms in equation (2.127) lead the linear (Kaiser, 1984) and nonlinear (Scoccimarro, 2004) redshift space power spectrum which has been studied before.

We find the 3rd order kernels of the galaxy density contrast in redshift space by substituting equation (2.73) into equation (2.127). The kernels are

$$K_1^{(s)}(\mathbf{k}) = c_1 \mathcal{W}_R(k) + f\mu^2 \quad (2.128)$$

$$\begin{aligned} K_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) = & \frac{c_2}{2} \mathcal{W}_R(q_1) \mathcal{W}_R(q_2) + c_1 \mathcal{W}_R(q_{12}) F_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) + f\mu^2 G_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \\ & + c_1 \frac{fk\mu}{2} \left[\frac{q_{1z}}{q_1^2} \mathcal{W}_R(q_2) + \frac{q_{2z}}{q_2^2} \mathcal{W}_R(q_1) \right] + \frac{(fk\mu)^2}{2} \frac{q_{1z}q_{2z}}{q_1^2 q_2^2} \end{aligned} \quad (2.129)$$

$$\begin{aligned} K_3^{(s)}(\mathbf{q}, -\mathbf{q}, \mathbf{k}) = & \frac{c_3}{6} \mathcal{W}_R^2(q) \mathcal{W}_R(k) + c_1 \mathcal{W}_R(k) F_3^{(s)}(\mathbf{q}, -\mathbf{q}, \mathbf{k}) \\ & + \frac{c_2}{3} \mathcal{W}_R(q) \left[\mathcal{W}_R(|\mathbf{k} + \mathbf{q}|) F_2^{(s)}(\mathbf{q}, \mathbf{k}) + \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) F_2^{(s)}(-\mathbf{q}, \mathbf{k}) \right] \\ & + f\mu^2 \left[G_3^{(s)}(\mathbf{q}, -\mathbf{q}, \mathbf{k}) + \frac{c_2}{6} \mathcal{W}_R^2(q) \right] \\ & + c_1 \frac{fk\mu}{3} \left[\mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) F_2^{(s)}(-\mathbf{q}, \mathbf{k}) \frac{q_z}{q^2} - \mathcal{W}_R(|\mathbf{k} + \mathbf{q}|) F_2^{(s)}(\mathbf{q}, \mathbf{k}) \frac{q_z}{q^2} \right. \\ & \quad \left. + \mathcal{W}_R(q) \left\{ G_2^{(s)}(-\mathbf{q}, \mathbf{k}) \frac{k_z - q_z}{|\mathbf{k} - \mathbf{q}|^2} + G_2^{(s)}(\mathbf{q}, \mathbf{k}) \frac{k_z + q_z}{|\mathbf{k} + \mathbf{q}|^2} \right\} \right] \\ & + \frac{(fk\mu)^2}{3} \left[-\frac{c_1}{2} \frac{q_z^2}{q^4} \mathcal{W}_R(k) + G_2^{(s)}(-\mathbf{q}, \mathbf{k}) \frac{q_z(k_z - q_z)}{q^2 |\mathbf{k} - \mathbf{q}|^2} \right. \\ & \quad \left. - G_2^{(s)}(\mathbf{q}, \mathbf{k}) \frac{q_z(k_z + q_z)}{q^2 |\mathbf{k} + \mathbf{q}|^2} \right] \\ & - \frac{(fk\mu)^3}{6} \frac{q_z^2 k_z}{q^4 k^2}. \end{aligned} \quad (2.130)$$

Note that we only show $K_3^{(s)}(\mathbf{q}, -\mathbf{q}, \mathbf{k})$, as it is what we need to calculate the third order power spectrum. The kernels in equation (2.128)~(2.130), when setting $\mathcal{W}_R \equiv 1$, coincide those shown in Heavens et al. (1998)¹⁸, and also reduces to the kernels for the redshift space matter density contrast, equation (2.110)~(2.112) when setting $c_1 = 1$ and $c_2 = c_3 = 0$. For $\mu = 0$, it reduces to the kernels for the galaxy density contrast in real space, as redshift space distortion does not affect the perpendicular directional wave modes.

As is the case for the mater power spectrum, we can reproduce the formulas widely used in the literature by taking the linear terms in equation (2.127):

$$\delta_{gs}(\mathbf{k}) = \delta_g(\mathbf{k}) + f\mu^2 \eta(\mathbf{k}). \quad (2.131)$$

¹⁸Except that the real space kernels, J_3 and K_3 , presented in Heavens et al. (1998) have to be replaced by $F_3^{(s)}$ s and $G_3^{(s)}$ s in equation (2.34) and equation (2.35), respectively.

By keeping only linear order of $\delta_g(\mathbf{k})$ and $\eta(\mathbf{k})$, and setting $\mathcal{W}_R(k) = 1$, we find the linear galaxy power spectrum in redshift space (Kaiser, 1987)

$$P_{gs}(k, \mu, z) = b^2(1 + \beta\mu^2)^2 D^2(z) P_L(k), \quad (2.132)$$

where b is the linear bias parameter and $\beta \equiv f/b$. Also, including third order density field and velocity field, while keeping bias linear, we find the formula used in Shoji et al. (2009):

$$P_{gs}(k, \mu, z) = b^2 [P_{\delta\delta}(k) + 2\beta\mu^2 P_{\delta\theta}(k) + \beta^2\mu^4 P_{\theta\theta}(k)], \quad (2.133)$$

where $P_{\delta\delta}(k)$, $P_{\delta\theta}(k)$ and $P_{\theta\theta}(k)$ are shown in equation (2.68), equation (2.117) and equation (2.118), respectively.

2.7.1 Gaussian case

Finally, we obtain the 3rd order galaxy power spectrum in redshift space by substituting these kernels into equation (2.58):

$$P_{gs}(k, z) = \langle \epsilon^2 \rangle + D^2(z) P_{gs,11}(k) + D^4(z) [P_{gs,22}(k) + 2P_{gs,13}(k)], \quad (2.134)$$

where

$$P_{gs,11}(k) = (c_1 \mathcal{W}_R(k) + f\mu^2)^2 P_L(k) \quad (2.135)$$

is the same as the linear galaxy power spectrum in the redshift space (linear redshift space distortion with linear bias, Kaiser, 1987), and non-linear terms are

$$\begin{aligned} P_{gs,22}(k) &= 2 \int \frac{d^3 q}{(2\pi)^3} P_L(q) P_L(|\mathbf{k} - \mathbf{q}|) \left[K_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \right]^2 \\ &= 2 \int \frac{d^3 q}{(2\pi)^3} P_L(q) P_L(|\mathbf{k} - \mathbf{q}|) \left[\left(f\mu^2 G_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \right)^2 \right. \\ &\quad + \left(\frac{c_2}{2} \mathcal{W}_R(q) \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) + c_1 F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \mathcal{W}_R(k) \right)^2 \\ &\quad + 2c_1 f\mu^2 \mathcal{W}_R(k) F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) G_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \left. \right] \\ &\quad + P_{gs,22}^{(rest)}(k, \mu; c_1, c_2), \end{aligned} \quad (2.136)$$

$$\begin{aligned}
P_{gs,13}(k, \mu) &= 3(c_1 \mathcal{W}_R(k) + f\mu^2)P_L(k) \int \frac{d^3q}{(2\pi)^3} P_L(q) K_3^{(s)}(\mathbf{q}, -\mathbf{q}, \mathbf{k}) \\
&= 3(c_1 \mathcal{W}_R(k) + f\mu^2)P_L(k) \int \frac{d^3q}{(2\pi)^3} P_L(q) \\
&\quad \times \left[\frac{c_3}{6} \mathcal{W}_R^2(q) \mathcal{W}_R(k) + c_1 \mathcal{W}_R(k) F_3^{(s)}(\mathbf{q}, -\mathbf{q}, \mathbf{k}) \right. \\
&\quad \left. + \frac{c_2}{3} \mathcal{W}_R(q) \left\{ \mathcal{W}_R(|\mathbf{k} + \mathbf{q}|) F_2^{(s)}(\mathbf{q}, \mathbf{k}) + \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) F_2^{(s)}(-\mathbf{q}, \mathbf{k}) \right\} \right. \\
&\quad \left. + f\mu^2 \left\{ G_3^{(s)}(\mathbf{q}, -\mathbf{q}, \mathbf{k}) + \frac{c_2}{6} \mathcal{W}_R^2(q) \right\} \right] + P_{gs,13}^{(rest)}(k, \mu; c_1). \tag{2.137}
\end{aligned}$$

Here, we only show terms which we can be explicitly identified as terms from non-linear evolution of matter ($P_{m,22}(k)$ or $P_{m,13}(k)$), from non-linear bias ($P_{b2}(k)$ or $P_{b22}(k)$), or from non-linear redshift space distortion ($P_{\theta\theta}(k)$ or $P_{\delta\theta}(k)$). Terms which cannot fall into those categories are called $P_{gs,22}^{(rest)}(k, \mu; c_1, c_2)$ and $P_{gs,13}^{(rest)}(k, \mu; c_1)$.

Combining equation (2.135) to (2.137), we calculate the galaxy power spectrum in redshift space

$$\begin{aligned}
P_{gs}(k, \mu, z) &= \langle \epsilon^2 \rangle + c_1^2 \mathcal{W}_R^2(k) P_{\delta\delta}(k, z) + D^2(z) [c_1 c_3 \sigma_R^2 + 8c_1 c_2 \sigma_R^2 \mathcal{G}_R(k)] \mathcal{W}_R^2(k) P_L(k, z) \\
&\quad + c_2^2 D^4(z) P_{b22}(k, z) + D^4(z) \frac{c_2^2}{2} \int \frac{d^3q}{(2\pi)^3} \mathcal{W}_R^2(q) P_L^2(q) + c_1 c_2 D^4(z) P_{b2}(k) \\
&\quad + 2f\mu^2 \left[c_1 \mathcal{W}_R(k) P_{\delta\theta}(k, z) + \frac{1}{2} D^2(z) \{ c_3 \sigma_R^2 + 8c_2 \sigma_R^2 \mathcal{G}_R(k) \} \mathcal{W}_R(k) P_L(k, z) \right. \\
&\quad \left. + \frac{1}{2} D^2(z) c_1 c_2 \sigma_R^2 \mathcal{W}_R(k) P_L(k) \right] + f^2 \mu^4 [P_{\theta\theta}(k, z) + c_2 D^2(z) \sigma_R^2 P_L(k, z)] \\
&\quad + D^4(z) [P_{gs,22}^{(rest)}(k, \mu; c_1, c_2) + 2P_{gs,13}^{(rest)}(k, \mu; c_1)], \tag{2.138}
\end{aligned}$$

where, $P_{\delta\delta}(k)$, $P_{b2}(k)$, $P_{b22}(k)$, $P_{\delta\theta}(k)$, $P_{\theta\theta}(k)$ are defined in equation (2.68), equation (2.87), equation (2.88), equation (2.117), and equation (2.118), respectively.

Following the discussion in Section 2.5, we shall absorb the small scale dependent quantities in equation (2.138) by re-defining the bias parameters: P_0 , b_1 and b_2 as is in equation (2.83), equation (2.84) and equation (2.85), respectively. In addition, we also introduce the new bias parameter b_θ :

$$b_\theta^2 = 1 + c_2 D^2(z) \sigma_R^2, \tag{2.139}$$

so that additional σ_R^2 term can be absorbed. By using the re-defined bias parameters, the

galaxy power spectrum in redshift space becomes

$$\begin{aligned}
P_{gs}(k, \mu, z) = & P_0 + b_1^2 [\mathcal{W}_R^2(k) P_m(k, z) + b_2 D^4(z) P_{b2}(k) + b_2^2 D^4(z) P_{b22}(k)] \\
& + 2f\mu^2 b_1 b_\theta \mathcal{W}_R(k) P_{\delta\theta}(k, z) + f^2 \mu^4 b_\theta^2 P_{\theta\theta}(k, z) \\
& + D^4(z) [P_{gs,22}^{(rest)}(k, \mu; b_1, b_2) + 2P_{gs,13}^{(rest)}(k, \mu; b_1)].
\end{aligned} \tag{2.140}$$

In quasi-nonlinear regime, r.m.s. density fluctuation is small $\sigma_R^2 \ll 1$; thus we make following approximations

$$b_1 \simeq c_1 + \frac{1}{2} D^2(z) [c_3 \sigma_R^2 + 8c_2 \sigma_R^2 \mathcal{G}_R(k)] \tag{2.141}$$

$$b_\theta \simeq 1 + \frac{c_2}{2} D^2(z) \sigma_R^2, \tag{2.142}$$

and

$$\sigma_R^2 P_L \simeq \sigma_R^2 P_{\delta\delta}(k) \simeq \sigma_R^2 P_{\delta\theta}(k) \simeq \sigma_R^2 P_{\theta\theta}(k),$$

which are true up to the fourth order in the linear density contrast. We also replace c_1 in the rest terms, $P_{gs,22}^{(rest)}(k, \mu, f; c_1, c_2)$ and $P_{gs,13}^{(rest)}(k, \mu, f; c_1)$, to b_1 , as it is consistent up to the same order. However, we set c_2 to be free, as this may allow us to check the consistency: $c_2 = b_1 b_2$.

Note that the new bias parameter b_θ is multiplied to the velocity divergence field, $\theta_{\mathbf{k}}$, thus we call it *velocity bias*. However, it does not mean that the velocity field itself is biased. The velocity bias is the bias introduced when ignoring the coupling between the density field and the velocity field in equation (2.127). For example, if one uses equation (2.133) as an estimator for the velocity power spectrum, then the measured velocity will be biased by at least a factor of $1 + b_\theta$, and the bias factor will increase when we also take ‘rest’ terms into account.

The ‘rest’ terms $P_{gs,22}^{(rest)}(k, \mu, f; c_1, c_2)$ $P_{gs,13}^{(rest)}(k, \mu, f; c_1)$ are defined as the collection of terms which cannot be simplified as either previously known terms or cannot be absorbed in the re-defined bias parameters. We show the explicit functional formula for the rest terms in Appendix D.

2.7.2 non-Gaussian case

We calculate the non-Gaussian correction to the galaxy power spectrum in redshift space as

$$\begin{aligned}
& \Delta P_g^s(k, \mu, z) \\
&= 2f_{\text{NL}} D^3(z) (c_1 \mathcal{W}_R(k) + f\mu^2) \mathcal{M}(k) \\
& \times \int \frac{d^3 q}{(2\pi)^3} \mathcal{M}(q) \mathcal{M}(|\mathbf{k} - \mathbf{q}|) P_\phi(q) [2P_\phi(k) + P_\phi(|\mathbf{k} - \mathbf{q}|)] \left[c_2 \mathcal{W}_R(q) \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) \right. \\
& \quad + 2c_1 \mathcal{W}_R(k) F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) + 2f\mu^2 G_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) + (fk\mu)^2 \frac{q_z(k\mu - q_z)}{q^2 |\mathbf{k} - \mathbf{q}|^2} \\
& \quad \left. + b_1(fk\mu) \left\{ \mathcal{W}_R(|\mathbf{k} - \mathbf{q}|) \frac{q_z}{q^2} + \mathcal{W}_R(q) \frac{k\mu - q_z}{|\mathbf{k} - \mathbf{q}|^2} \right\} \right]. \tag{2.143}
\end{aligned}$$

We, again, find that the non-Gaussianity term comes from the non-linear redshift space mapping is parametrically small, and the dominant term is

$$\begin{aligned}
\Delta P_g^s(k, \mu, z) &\simeq 2f_{\text{NL}} D^3(z) (c_1 \mathcal{W}_R(k) + f\mu^2) \mathcal{M}(k) \\
& \times c_2 \int \frac{d^3 q}{(2\pi)^3} \mathcal{M}_R(q) \mathcal{M}_R(|\mathbf{k} - \mathbf{q}|) P_\phi(q) [2P_\phi(k) + P_\phi(|\mathbf{k} - \mathbf{q}|)] \\
& = 4f_{\text{NL}} D^3(z) (c_1 \mathcal{W}_R(k) + f\mu^2) c_2 \sigma_R^2 \mathcal{F}_R(k) \frac{P_L(k)}{\mathcal{M}(k)}. \tag{2.144}
\end{aligned}$$

By using $\tilde{b}_2 \equiv \sigma_R^2 D^2(z) c_2 / c_1$ we define in Section 2.5.3, the correction term becomes

$$\Delta P_g^s(k, \mu, z) = 4f_{\text{NL}} D(z) c_1 (c_1 \mathcal{W}_R(k) + f\mu^2) \tilde{b}_2 \mathcal{F}_R(k) \frac{P_L(k)}{\mathcal{M}(k)}. \tag{2.145}$$

On large scales ($k \ll 1/R$), $\mathcal{F}_R(k) = \mathcal{W}_R(k) = 1$, and for the highly biased tracers, $\tilde{b}_2 \simeq \alpha \delta_c^{19}$, we can rewrite the non-Gaussian term as

$$\Delta P_g^s(k, \mu, z) = 6\alpha \delta_c f_{\text{NL}} D(z) c_1 (c_1 + f\mu^2) \frac{H_0^2 \Omega_m P_L(k)}{k^2 T(k)}. \tag{2.146}$$

Therefore, on large scale, the non-Gaussian galaxy power spectrum in redshift space is given by

$$P_{gs}(k, \mu, z) = [b_1 + f\mu^2]^2 P_L(k) + 6\alpha f_{\text{NL}} \delta_c b_1 (b_1 + f\mu^2) \frac{H_0^2 \Omega_m P_L(k)}{k^2 T(k)} + P_0. \tag{2.147}$$

Here, we approximate $b_1 \simeq c_1$ as the equation is written in the linear order.

¹⁹ $\alpha = 1$ for Press-Schechter mass function, and $\alpha = 0.75$ for Sheth-Tormen mass function. For more discussion, see Section 2.5.3.

2.8 Summary

In this chapter, we calculate the nonlinear galaxy power spectrum in redshift space within Eulerian perturbation theory framework. After defining three conditions in the quasi non-linear regime, 1) sub-Horizon, 2) pressureless, 3) curl-free, we find the perturbative solutions for the non-linear evolution of the density field and velocity field of cosmic matter field. By using the perturbation theory solution, we calculate the matter power spectrum and the galaxy power spectrum in both real and redshift space. For each case, we also calculate the leading order correction to the power spectrum from the local primordial non-Gaussianity.

2.8.1 On the smoothing of density field

When applying the locality of bias assumption, we assume that the galaxy density contrast is a local function of the smoothed matter density contrast. While it is not certain if locality of bias works for the *real* density contrast or the smoothed density contrast, we intentionally adopt the smoothing function as it facilitates to access the effect of the small scale smoothing on the large scale power spectrum.

As we are mainly interested in the power spectrum on sufficiently large scales, $k \ll 1/R$, we want to absorb the effect of small scale smoothing into the value of the *free* bias parameters; thus, our theoretical template of the galaxy power spectrum in redshift space should not depend on the smoothing scale R . We test that every term in equation (2.140) is indeed independent on the smoothing scale as we desire.

In short, in the light of the re-definition of bias parameters, the smoothing does not affect the power spectrum on large scales; smoothing scale (or halo mass) will only changes the exact value of the bias parameters. Therefore, we show the unsmoothed equations in the summary below:

2.8.2 Summary of equations

The galaxy power spectrum in redshift space on the quasi-nonlinear scales is given by

$$\begin{aligned}
 P_{gs}(k, \mu, z) = & P_0 + b_1^2 [P_m(k, z) + b_2 D^4(z) P_{b2}(k) + b_2^2 D^4(z) P_{b22}(k)] \\
 & + 2f\mu^2 b_1 b_\theta P_{\delta\theta}(k, z) + f^2 \mu^4 b_\theta^2 P_{\theta\theta}(k, z) \\
 & + D^4(z) [P_{gs,22}^{(rest)}(k, \mu; b_1, b_2) + 2P_{gs,13}^{(rest)}(k, \mu; b_1)], \quad (2.148)
 \end{aligned}$$

where P_0 , b_1 , b_2 , b_θ , c_2 are free bias parameters, and the components of nonlinear power spectrum are listed as follow.

$$P_m(k, z) = D^2(z) [P_L(k) + D^2(z) \{P_{m,22}(k) + 2P_{m,13}(k)\}] \quad (2.149)$$

$$P_{b2}(k) = 2 \int \frac{d^3q}{(2\pi)^3} P_L(q) P_L(|\mathbf{k} - \mathbf{q}|) F_2^{(s)}(\mathbf{q}, \mathbf{k} - \mathbf{q}) \quad (2.150)$$

$$P_{b22}(k) = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} P_L(q) [P_L(|\mathbf{k} - \mathbf{q}|) - P_L(q)] \quad (2.151)$$

$$P_{\delta\theta}(k, z) = D^2(z) [P_L(k) + D^2(z) \{P_{\delta\theta,22}(k) + P_{\delta\delta,13}(k) + P_{\theta\theta,13}(k)\}] \quad (2.152)$$

$$P_{\theta\theta}(k, z) = D^2(z) [P_L(k) + D^2(z) \{P_{\theta\theta,22}(k) + 2P_{\theta\theta,13}(k)\}] \quad (2.153)$$

The P_{22} and P_{13} terms in P_m , $P_{\delta\theta}$, $P_{\theta\theta}$ are

$$P_{m,22}(k) = \frac{1}{98} \frac{k^3}{(2\pi)^2} \int dr P_L(kr) \int_{-1}^1 dx P_L(k\sqrt{1+r^2-2rx}) \left[\frac{7x+3r-10rx^2}{1+r^2-2rx} \right]^2 \quad (2.154)$$

$$P_{m,13}(k) = \frac{1}{504} \frac{k^3}{(2\pi)^2} P_L(k) \int dr P_L(kr) \times \left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^3} (r^2 - 1)^3 (7r^2 + 2) \ln \left(\frac{r+1}{|r-1|} \right) \right] \quad (2.155)$$

$$P_{\delta\theta,22}(k) = \frac{1}{98} \frac{k^3}{(2\pi)^2} \int dr P_L(kr) \int_{-1}^1 dx P_L(k\sqrt{1+r^2-2rx}) \times \frac{(7x+3r-10rx^2)(7x-r-6rx^2)}{(1+r^2-2rx)^2} \quad (2.156)$$

$$P_{\theta\theta,22}(k) = \frac{1}{98} \frac{k^3}{(2\pi)^2} \int dr P_L(kr) \int_{-1}^1 dx P_L(k\sqrt{1+r^2-2rx}) \left[\frac{7x-r-6rx^2}{1+r^2-2rx} \right]^2 \quad (2.157)$$

$$P_{\theta\theta,13}(k) = \frac{1}{504} \frac{k^3}{(2\pi)^2} P_L(k) \int dr P_L(kr) \times \left[\frac{36}{r^2} - 246 + 12r^2 - 18r^4 + \frac{9}{r^3} (r^2 - 1)^3 (r^2 + 2) \ln \left(\frac{1+r}{|1-r|} \right) \right], \quad (2.158)$$

and the ‘rest’ terms $P_{gs,22}^{(rest)}(k, \mu, f; b_1, c_2)$ $P_{gs,13}^{(rest)}(k, \mu, f; b_1)$ are shown in Appendix D (except that now we set all smoothing function to be unity; $\mathcal{W}_R = 1$).

By setting appropriate bias parameters and an angle parameter μ , one can reproduce the result for different cases we study in this chapter. If setting $P_0 = 0$ $b_1 = 1$, $b_2 = 0$, $b_\theta = 0$, equation (2.148) reduces to the non-linear matter power spectrum in redshift space, whose $\mu = 0$ slice is the non-linear matter power spectrum in real space, and, for $b_\theta = 0$, $\mu = 0$, equation (2.148) reduces to the non-linear galaxy power spectrum in real space.

Finally, on large scales, the dominant correction of galaxy power spectrum due to primordial local non-Gaussianity is given by

$$\Delta P_g^s(k, \mu, z) = 6\alpha\delta_c f_{\text{NL}} D(z) b_1 (b_1 + f\mu^2) \frac{H_0^2 \Omega_m P_L(k)}{k^2 T(k)}. \quad (2.159)$$

Appendix E

Lagrangian Perturbation Theory and Initial condition for Cosmological N-body Simulation

We generate the initial condition for cosmological N-body simulation by using Lagrangian perturbation theory. In this section, we review the linear and second order Lagrangian perturbation theory and compare the cosmological initial condition generated from two theories.

E.1 Lagrangian perturbation theory formalism

Let us summarize the result of the Lagrangian perturbation theory. The reader can find a review on the subject in Bouchet et al. (1995); Bernardeau et al. (2002).

While Eulerian perturbation theory (Chapter 2) describes the density and velocity fields of matter at a fixed (‘comoving’ in cosmology) coordinate system, Lagrangian perturbation theory concentrates on the trajectory of individual particle. We denote the Eulerian (comoving physical) coordinate $\mathbf{x}$, and the Lagrangian (comoving initial) coordinate $\mathbf{q}$. As we define the both coordinate in comoving sense, the expansion of Universe does not change them. In Lagrangian perturbation theory, the dynamical variable is the Lagrangian displacement field $\Psi(\mathbf{q}, \tau)$, which is defined by

$$\mathbf{x}(\tau) = \mathbf{q} + \Psi(\mathbf{q}, \tau). \quad (\text{E.1})$$

Note that $\Psi = 0$ initially so that $\mathbf{q}$ is the same as the usual comoving coordinate at initial time, $\tau = 0$.

The particle trajectory in the expanding universe is governed by the equation of motion:

$$\frac{d^2 \mathbf{x}}{d\tau^2} + \mathcal{H}(\tau) \frac{d\mathbf{x}}{d\tau} = -\nabla_{\mathbf{x}} \Phi, \quad (\text{E.2})$$

where Φ is the peculiar gravitational potential, and

$$\mathcal{H}(\tau) \equiv \frac{1}{a} \frac{da}{d\tau} = a(t)H(t) \quad (\text{E.3})$$

is the modified Hubble parameter. Note that we are using a conformal time τ which is related to the Robertson-Walker coordinate time by $dt = a d\tau$. Taking a divergence of this equation, we get

$$\nabla_{\mathbf{x}} \cdot \left[\frac{d^2 \mathbf{x}}{d\tau^2} + \mathcal{H}(\tau) \frac{d\mathbf{x}}{d\tau} \right] = -\nabla_{\mathbf{x}}^2 \Phi = -\frac{3}{2} \mathcal{H}^2 \Omega_m \delta(\mathbf{x}), \quad (\text{E.4})$$

where $\delta(\mathbf{x})$ is the density contrast, $\delta(\mathbf{x}) \equiv \rho(\mathbf{x})/\bar{\rho} - 1$.

The particle density in the Lagrangian coordinate is the same as the average density of the universe. Therefore, by using the mass conservation, we have

$$\bar{\rho}(\tau) d^3 q = \rho(\mathbf{x}, \tau) d^3 x = \bar{\rho}(\tau) [1 + \delta(\mathbf{x}, \tau)] d^3 x. \quad (\text{E.5})$$

By using the equation above, we can relate the Eulerian density contrast $\delta(\mathbf{x}, \tau)$ to the Lagrangian displacement vector $\Psi(\mathbf{q}, \tau)$ as

$$1 + \delta(\mathbf{x}, \tau) = \left| \frac{d^3 q}{d^3 x} \right| = \frac{1}{J(\mathbf{q}, \tau)}, \quad (\text{E.6})$$

where

$$J(\mathbf{q}, \tau) = \det(\delta_{ij} + \Psi_{i,j}(\mathbf{q}, \tau)), \quad (\text{E.7})$$

is a Jacobian of the Lagrangian to Eulerian coordinate transform. Here, we abbreviate the partial derivative with respect q_j coordinate as $\Psi_{i,j} \equiv \partial \Psi_i / \partial q_j$.

By using equation (E.6), the equation of motion becomes

$$J(\mathbf{q}, \tau) \nabla_{\mathbf{x}} \cdot \left[\frac{d^2 \mathbf{x}}{d\tau^2} + \mathcal{H}(\tau) \frac{d\mathbf{x}}{d\tau} \right] = \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) (J - 1). \quad (\text{E.8})$$

Using the chain rule

$$\frac{\partial}{\partial x_i} = \left[\frac{d^3 q}{d^3 x} \right]_{ij} \frac{\partial}{\partial q_j} = [\delta_{ij} + \Psi_{i,j}]^{-1} \frac{\partial}{\partial q_j},$$

the equation for displacement vector Ψ becomes

$$\begin{aligned} J(\mathbf{q}, \tau) [\delta_{ij} + \Psi_{i,j}(\mathbf{q}, \tau)]^{-1} \left[\frac{d^2 \Psi_{i,j}(\mathbf{q}, \tau)}{d\tau^2} + \mathcal{H}(\tau) \frac{d\Psi_{i,j}(\mathbf{q}, \tau)}{d\tau} \right] \\ = \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) [J(\mathbf{q}, \tau) - 1]. \end{aligned} \quad (\text{E.9})$$

Equation (E.9) is the master equation of the Lagrangian perturbation theory. In order to get the perturbative solution, we solve the equation perturbatively in $\Psi(\mathbf{q}, \tau)$:

$$\Psi(\mathbf{q}, \tau) = \Psi^{(1)}(\mathbf{q}, \tau) + \Psi^{(2)}(\mathbf{q}, \tau) + \dots \quad (\text{E.10})$$

E.2 Linear Lagrangian perturbation theory

In order to get the linear solution, let's first approximate the Jacobian as

$$J(\mathbf{q}, \tau) = \det[\delta_{ij} + \Psi_{i,j}(\mathbf{q}, \tau)] \simeq 1 + \Psi_{i,i}(\mathbf{q}, \tau), \quad (\text{E.11})$$

and the inverse of the Jacobian matrix as

$$[\delta_{ij} + \Psi_{i,j}]^{-1} \simeq \delta_{ij} - \Psi_{i,j}. \quad (\text{E.12})$$

This approximation is justified by using the matrix approximation of $\det(I + A) = 1 + \text{tr}(A) + \mathcal{O}(A^2)^1$. Using these linear approximations, equation (E.9) becomes

$$(1 + \Psi_{k,k})[\delta_{ij} - \Psi_{i,j}] \left(\frac{d^2 \Psi_{i,j}}{d\tau^2} + \mathcal{H}(\tau) \frac{d\Psi_{i,j}}{d\tau} \right) = \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) \Psi_{k,k}, \quad (\text{E.15})$$

and we can find the linear equation for $\Psi^{(1)}$:

$$\frac{d^2 \Psi_{i,i}^{(1)}}{d\tau^2} + \mathcal{H}(\tau) \frac{d\Psi_{i,i}^{(1)}}{d\tau} = \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) \Psi_{i,i}^{(1)}. \quad (\text{E.16})$$

Since the dynamical variable of the equation of motion is only time, we can separate the time dependent part as $D_1(\tau)$. Also, in linear approximation,

$$1 + \delta_1(\mathbf{x}, \tau) = J^{-1} \simeq 1 - \Psi_{i,i},$$

therefore, the linear solution becomes

$$\nabla_q \cdot \Psi^{(1)} = -\delta_1(\mathbf{x}, \tau), \quad (\text{E.17})$$

where the time evolution of $\delta_1(\mathbf{x}, \tau)$ is governed by following equation.

$$D_1''(\tau) + \mathcal{H}(\tau) D_1'(\tau) = \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) D_1(\tau). \quad (\text{E.18})$$

¹This identity can be proved as follows. Suppose we have a matrix A , whose eigenvalues are small. From the matrix identity

$$\det(C) = \exp[\text{tr}(\ln C)], \quad (\text{E.13})$$

and when $C = I + A$, where I is the identity matrix, we get

$$\begin{aligned} \det(I + A) = \exp[\text{tr}(\ln(I + A))] &= \exp \left[\text{tr} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} A^n}{n} \right) \right] \\ &= 1 + \text{tr} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} A^n}{n} \right) + \text{tr} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} A^n}{n} \right)^2 + \dots \\ &= 1 + \text{tr}(A) + \frac{1}{2} [\text{tr}^2(A) - \text{tr}(A^2)] + \mathcal{O}(A^3). \end{aligned} \quad (\text{E.14})$$

Note that the linear growth function $D_1(\tau)$ here is the same as the linear growth factor in Eulerian perturbation theory [Eq. (2.11)].

Therefore, the particle position in the linear Lagrangian perturbation theory is given by

$$\mathbf{x} = \mathbf{q} - \nabla_q^{-1} \delta_1(\mathbf{x}, \tau), \quad (\text{E.19})$$

and corresponding peculiar velocity is

$$\mathbf{v} \equiv \frac{d\mathbf{x}}{d\tau} = -\mathcal{H} f_1 \nabla_q^{-1} \delta_1(\mathbf{x}, \tau). \quad (\text{E.20})$$

Here, $f_1 \equiv d \ln D_1 / d \ln a$ is a logarithmic derivative of the linear growth factor.

We can further simplify the result by using the curl-free condition of the Lagrangian displacement. The ir-rotational condition $\nabla \times \Psi^{(1)} = 0$ implies the existence of a scalar potential satisfying $\Psi^{(1)}(\mathbf{q}, \tau) = -\nabla_q \phi^{(1)}(\mathbf{q}, \tau)$. Therefore, in terms of the linear Lagrangian potential $\phi^{(1)}$, which is related to the linear density field as

$$\nabla_q \cdot \Psi^{(1)}(\mathbf{q}, \tau) = -\nabla_q^2 \phi^{(1)}(\mathbf{q}, \tau) = -\delta_1(\mathbf{x}, \tau), \quad (\text{E.21})$$

the linear solution can be also written as

$$\mathbf{x}(\mathbf{q}, \tau) = \mathbf{q} - \nabla_q \phi^{(1)}(\mathbf{q}, \tau) \quad (\text{E.22})$$

$$\mathbf{v}(\mathbf{q}, \tau) = -\mathcal{H} f_1 \nabla_q \phi^{(1)}(\mathbf{q}, \tau). \quad (\text{E.23})$$

E.2.1 Zel'dovich approximation

The Zel'dovich approximation (Zel'dovich, 1970) extrapolates the linear solution [Eq. (E.22) and Eq. (E.23)] into the non-linear regime. As a consequence of such an approximation, the theory predict what's called *Zel'dovich pancake*.

In this context, the Jacobian matrix is often called the ‘tensor of deformation’:

$$\mathfrak{D}_{ij} \equiv \delta_{ij} + \Psi_{i,j}(\mathbf{q}, \tau). \quad (\text{E.24})$$

We can find the three eigenvectors of the deformation tensor $\mathfrak{D}_{ij}$ and using these eigensystem as a basis (with eigenvalues $-\alpha$, $-\beta$, and $-\gamma$ ²), we can diagonalize³ it as

$$J(\mathbf{q}, \tau) = \left\| \begin{array}{ccc} 1 - \alpha D_1(\tau) & 0 & 0 \\ 0 & 1 - \beta D_1(\tau) & 0 \\ 0 & 0 & 1 - \gamma D_1(\tau) \end{array} \right\|, \quad (\text{E.25})$$

²Negative eigenvalues correspond to the growing mode (positive $\delta(\mathbf{x}, \tau)$) See, for example, equation (E.17).

³This diagonalization is justified because, we assume the ir-rotational perturbation; using a scalar potential $\phi^{(1)}$, $\mathfrak{D}_{ij}$ is symmetric (i.e. diagonalizable).

where $D_1(\tau)$ is the linear growth function. By using the three eigenvalues, we can write the density contrast as [Eq. (E.6)]

$$\delta(\mathbf{x}, \tau) = [(1 - D_1(\tau)\alpha)(1 - D_1(\tau)\beta)(1 - D_1(\tau)\gamma)]^{-1} - 1. \quad (\text{E.26})$$

What is the consequence of it? In order to see that, without loss of generality, let α be the largest eigenvalues of deformation tensor. For example, for the ellipsoidal shape initial perturbation, α corresponds to the shortest axis. Initially, $D_1(\tau)$ is small, but it gets bigger as linear perturbation grows, and eventually, reaches $1/\alpha$. At that moment, the density of α -direction become infinity, and this stage is called *Zeldovich pancake*. Physically, it mean that every sheet element sliced perpendicular to α -direction finally merging at one point. In reality, however, Zeldovich approximation breaks down before the volume element reaches its infinite density stage. It is because for the particles within such a small volume element, we have to take into account the pressure effect as well as shell crossing.

E.3 Second order Lagrangian perturbation theory (2LPT)

Up to second order, the Jacobian is approximated as [Eq. (E.14)]

$$J \simeq 1 + \Psi_{k,k}^{(1)} + \Psi_{k,k}^{(2)} + \frac{1}{2} \left[\left(\Psi_{k,k}^{(1)} \right)^2 - \Psi_{i,j}^{(1)} \Psi_{j,i}^{(1)} \right], \quad (\text{E.27})$$

therefore equation of motion,

$$J [\delta_{ij} - \Psi_{i,j}] \left(\frac{d^2 \Psi_{i,j}}{d\tau^2} + \mathcal{H} \frac{d\Psi_{i,j}}{d\tau} \right) = \frac{3}{2} \mathcal{H}^2 \Omega_m (J - 1), \quad (\text{E.28})$$

becomes

$$\begin{aligned} & \left(\frac{d^2 \Psi_{i,i}^{(2)}}{d\tau^2} + \mathcal{H} \frac{d\Psi_{i,i}^{(2)}}{d\tau} \right) + \Psi_{k,k}^{(1)} \left(\frac{d^2 \Psi_{i,i}^{(1)}}{d\tau^2} + \mathcal{H} \frac{d\Psi_{i,i}^{(1)}}{d\tau} \right) - \Psi_{i,j}^{(1)} \left(\frac{d^2 \Psi_{i,j}^{(1)}}{d\tau^2} + \mathcal{H} \frac{d\Psi_{i,j}^{(1)}}{d\tau} \right) \\ &= \frac{3}{2} \mathcal{H}^2 \Omega_m \left[\Psi_{k,k}^{(2)} + \frac{1}{2} \left(\Psi_{k,k}^{(1)} \right)^2 - \frac{1}{2} \Psi_{i,j}^{(1)} \Psi_{j,i}^{(1)} \right]. \end{aligned} \quad (\text{E.29})$$

Using the solution for the linear displacement field $\Psi^{(1)}(\mathbf{x}, \tau)$, we simplify the equation of motion as

$$\left(\frac{d^2 \Psi_{i,i}^{(2)}}{d\tau^2} + \mathcal{H} \frac{d\Psi_{i,i}^{(2)}}{d\tau} - \frac{3}{2} \mathcal{H}^2 \Omega_m \Psi_{i,i}^{(2)} \right) = -\frac{3}{2} \mathcal{H}^2 \Omega_m \left[\frac{1}{2} \left(\Psi_{k,k}^{(1)} \right)^2 - \frac{1}{2} \Psi_{i,j}^{(1)} \Psi_{j,i}^{(1)} \right]. \quad (\text{E.30})$$

Here, we use the following symmetry: $\Psi_{i,j}^{(1)} = \Psi_{j,i}^{(1)} = -\phi_{,ij}^{(1)}$. The equation of motion above is, again, separable, because the spatial derivatives only appears as divergence. Let's denote

the time dependent part of $\Psi^{(2)}$ as $D_2(\tau)$. The second order time evolution of $D_2(\tau)$ is governed by

$$D_2''(\tau) + \mathcal{H}D_2'(\tau) - \frac{3}{2}\mathcal{H}^2\Omega_m D_2(\tau) = -\frac{3}{2}\mathcal{H}^2\Omega_m [D_1(\tau)]^2. \quad (\text{E.31})$$

Note that, in flat Λ CDM universe, $D_2(\tau) \simeq -3D_1^2(\tau)\Omega_m^{-1/143}/7$ approximates $D_2(\tau)$ better than 0.6 per cent (Bouchet et al. 1995).

The space part of the second order solution describes the effect of gravitational tide,

$$\frac{\Psi_{k,k}^{(2)}(\mathbf{q}, \tau)}{D_2(\tau)} = \frac{1}{D_1^2(\tau)} \frac{1}{2} \sum_{i \neq j} \left\{ \Psi_{i,i}^{(1)}(\mathbf{q}, \tau) \Psi_{j,j}^{(1)}(\mathbf{q}, \tau) - \Psi_{i,j}^{(1)}(\mathbf{q}, \tau) \Psi_{j,i}^{(1)}(\mathbf{q}, \tau) \right\}. \quad (\text{E.32})$$

By using a second order scalar potential $\Psi^{(2)}(\mathbf{q}, \tau) = \nabla_{\mathbf{q}} \phi^{(2)}(\mathbf{q}, \tau)$, the equation becomes

$$\nabla_{\mathbf{q}}^2 \phi^{(2)}(\mathbf{q}, \tau) = \frac{D_2(\tau)}{D_1^2(\tau)} \frac{1}{2} \sum_{i \neq j} \left\{ \phi_{,ii}^{(1)}(\mathbf{q}, \tau) \phi_{,jj}^{(1)}(\mathbf{q}, \tau) - \phi_{,ij}^{(1)}(\mathbf{q}, \tau) \phi_{,ji}^{(1)}(\mathbf{q}, \tau) \right\} \quad (\text{E.33})$$

$$\simeq -\frac{3}{7}\Omega_m^{-1/143} \sum_{i > j} \left\{ \phi_{,ii}^{(1)}(\mathbf{q}, \tau) \phi_{,jj}^{(1)}(\mathbf{q}, \tau) - \left[\phi_{,ij}^{(1)}(\mathbf{q}, \tau) \right]^2 \right\}. \quad (\text{E.34})$$

Using the scalar potential, the solution for the position and velocity up to the second order are

$$\mathbf{x}(\mathbf{q}, \tau) = \mathbf{q} - \nabla_{\mathbf{q}} \phi^{(1)}(\mathbf{q}, \tau) + \nabla_{\mathbf{q}} \phi^{(2)}(\mathbf{q}, \tau) \quad (\text{E.35})$$

$$\mathbf{v}(\mathbf{q}, \tau) = \frac{d\mathbf{x}}{d\tau} = -\mathcal{H}f_1 \nabla_{\mathbf{q}} \phi^{(1)}(\mathbf{q}, \tau) + \mathcal{H}f_2 \nabla_{\mathbf{q}} \phi^{(2)}(\mathbf{q}, \tau). \quad (\text{E.36})$$

The logarithmic derivatives of the growth factor $f_i \equiv d \ln D_i / d \ln a$ is well approximated as

$$f_1 \simeq [\Omega_m(z)]^{5/9}, \quad f_2 \simeq 2 [\Omega_m(z)]^{6/11}$$

to better than 10 and 12 percent, respectively, for flat- Λ CDM universe with $0.01 < \Omega_m < 1$. The accuracy of these two fits improves significantly for $\Omega_m \geq 0.1$ (Bouchet et al. 1995).

E.4 Generating initial condition using Linear solution

In this section, we present the way how to generate the initial condition by using the linear, and second order Lagrangian perturbation theory solutions. For the detailed normalization, and the way to impose the Hermitian condition in Fourier space, see, Appendix A.

The solutions of Lagrangian perturbation theory is summarized as below: The second order particle position and velocity are given by

$$\mathbf{x}(\mathbf{q}, \tau) = \mathbf{q} - \nabla_{\mathbf{q}}\phi^{(1)}(\mathbf{q}, \tau) + \nabla_{\mathbf{q}}\phi^{(2)}(\mathbf{q}, \tau) \quad (\text{E.37})$$

$$\mathbf{v}(\mathbf{q}, \tau) = \frac{d\mathbf{x}}{d\tau} = -\mathcal{H}f_1\nabla_{\mathbf{q}}\phi^{(1)}(\mathbf{q}, \tau) + \mathcal{H}f_2\nabla_{\mathbf{q}}\phi^{(2)}(\mathbf{q}, \tau), \quad (\text{E.38})$$

where the linear and the second order Lagrangian potential is

$$\nabla_{\mathbf{q}}^2\phi^{(1)}(\mathbf{q}, \tau) = \delta_1(\mathbf{x}, \tau), \quad (\text{E.39})$$

and

$$\nabla_{\mathbf{q}}^2\phi^{(2)}(\mathbf{q}, \tau) \simeq -\frac{3}{7}[\Omega_m(\tau)]^{-1/143} \sum_{i>j} \left\{ \phi_{,ii}^{(1)}(\mathbf{q}, \tau)\phi_{,jj}^{(1)}(\mathbf{q}, \tau) - \left[\phi_{,ij}^{(1)}(\mathbf{q}, \tau) \right]^2 \right\}, \quad (\text{E.40})$$

respectively. Here, $\mathcal{H}(\tau)$ is a reduced Hubble parameter

$$\mathcal{H}(z) = \frac{H_0}{1+z} \sqrt{\Omega_\Lambda + \Omega_m(1+z)^3}, \quad (\text{E.41})$$

and $f_i \equiv d \ln D_i / d \ln a$ is the logarithmic derivative of growth factor, which can be approximated as

$$f_1 \simeq [\Omega_m(z)]^{5/9}, \quad f_2 \simeq 2[\Omega_m(z)]^{6/11}, \quad (\text{E.42})$$

for flat- Λ CDM universe. For different cosmology, one has to solve the linear and non-linear growth equations: [Eq. (E.18)] and [Eq. (E.31)].

Let us suppose that we want to generate the initial condition for N^3 matter particles inside of cubic box of volume L^3 at redshift z . The initial density field follows the Gaussian statistics, and the linear power spectrum is given $P_L(k, z)$. Then, the procedure is as follow.

- (1) Imagine we divide the cubic box into N^3 regular grid points. Those are the Lagrangian coordinate $\mathbf{q}$.
- (2) As Lagrangian displacement vector is determined by the density contrast, we need to generate the Gaussian random density contrast. From the definition of power spectrum

$$\langle \delta_1(\mathbf{k}, z) \delta_1(\mathbf{k}', z) \rangle = (2\pi)^3 P_L(k, z) \delta^D(\mathbf{k} + \mathbf{k}'),$$

we relate the density contrast to the power spectrum as

$$\langle |\delta(\mathbf{k}, z)|^2 \rangle = \frac{(2\pi)^3}{k_F^3} P(k, z) = V P(k, z), \quad (\text{E.43})$$

where we use $\delta^D(\mathbf{k} + \mathbf{k}') = \delta_{\mathbf{n}_k, \mathbf{n}_{k'}}^k / k_F^3$ with the fundamental frequency $k_F = (2\pi)/L$. Here, δ_{ij}^k is the Kronecker delta.

Generate density contrast in the Fourier space $\delta(\mathbf{k}) \equiv \delta_r(\mathbf{k}) + i\delta_i(\mathbf{k})$ as random variables obeying a Gaussian statistics with mean and variance given by:

$$\langle \delta_r(\mathbf{k}) \rangle = 0, \quad \langle \delta_r^2(\mathbf{k}) \rangle = \sqrt{\frac{VP(k)}{2}} \quad (\text{E.44})$$

$$\langle \delta_i(\mathbf{k}) \rangle = 0, \quad \langle \delta_i^2(\mathbf{k}) \rangle = \sqrt{\frac{VP(k)}{2}}. \quad (\text{E.45})$$

When generating the random variable, we have to explicitly impose the Hermitian condition of $\delta(\mathbf{k})$:

$$\delta(-\mathbf{k}) = \delta^*(\mathbf{k}). \quad (\text{E.46})$$

If one want to calculate the real space density field, do the inverse Fourier transform by

$$\delta(\mathbf{q}) = \frac{1}{V} \sum \delta(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{q}} = \frac{1}{V} \delta^{\text{FFTW}}(\mathbf{q}). \quad (\text{E.47})$$

(3) The linear Lagrangian potential $\phi^{(1)}$ is given by

$$-k^2 \phi^{(1)}(\mathbf{k}, z) = \delta_1(\mathbf{k}, z). \quad (\text{E.48})$$

Therefore, calculate the linear Lagrangian displacement $\Psi^{(1)}$ in Fourier space by

$$\Psi^{(1)}(\mathbf{k}, z) = -i\mathbf{k} \phi^{(1)}(\mathbf{k}, z) = i\mathbf{k} \frac{\delta_1(\mathbf{k}, z)}{k^2}. \quad (\text{E.49})$$

(4) Inverse Fourier transform to get the displacement field in real space.

$$\Psi^{(1)}(\mathbf{q}, z) = \frac{1}{V} \sum i\mathbf{k} \frac{\delta_1(\mathbf{k}, z)}{k^2} e^{i\mathbf{k} \cdot \mathbf{q}}. \quad (\text{E.50})$$

Then, move particles at each grid point $\mathbf{q}$ by the displacement vector at that point.

$$\mathbf{x} = \mathbf{q} + \Psi^{(1)}(\mathbf{q}, z) \quad (\text{E.51})$$

Also, assign the velocity as

$$\mathbf{v} = f_1(z) \mathcal{H}(z) \Psi^{(1)}(\mathbf{q}, z). \quad (\text{E.52})$$

Now, we have generated the Zeldovich initial condition.

- (5) In order to calculate the initial condition by using second order Lagrangian perturbation theory (2LPT), we have to calculate the second order Lagrangian potential $\phi^{(2)}$ from equation (E.40). We shall solve this equation in Fourier space with following order.

- (5)-1 Calculate the Fourier transform of $\phi_{,ij}^{(1)}(\mathbf{q}, \tau)$ in Fourier space:

$$-k_i k_j \phi^{(1)}(\mathbf{k}, \tau) = \frac{k_i k_j \delta_1(\mathbf{k}, \tau)}{k^2}. \quad (\text{E.53})$$

- (5)-2 Inverse Fourier transform to the real space, now we have six $\phi_{,ij}^{(1)}(\mathbf{q}, z)$ s. Calculate the right hand side of equation (E.40) as

$$F(\mathbf{q}) = \left[\phi_{,22}^{(1)}(\mathbf{q}, z) \phi_{,11}^{(1)}(\mathbf{q}, z) + \phi_{,33}^{(1)}(\mathbf{q}, z) \phi_{,22}^{(1)}(\mathbf{q}, z) + \phi_{,33}^{(1)}(\mathbf{q}, z) \phi_{,11}^{(1)}(\mathbf{q}, z) \right. \\ \left. - \left[\phi_{,32}^{(1)}(\mathbf{q}, z) \right]^2 - \left[\phi_{,31}^{(1)}(\mathbf{q}, z) \right]^2 - \left[\phi_{,21}^{(1)}(\mathbf{q}, z) \right]^2 \right]. \quad (\text{E.54})$$

- (5)-3 Fourier transform $F(\mathbf{q})$. Then, $\phi^{(2)}(\mathbf{k}, z)$ is

$$\phi^{(2)}(\mathbf{k}, z) = \frac{3}{7} [\Omega_m(z)]^{-1/143} \frac{F(\mathbf{k}, z)}{k^2}, \quad (\text{E.55})$$

and we calculate the second order Lagrangian displacement $\Psi^{(2)}$ by

$$\Psi^{(2)}(\mathbf{k}, z) = i\mathbf{k}\phi^{(2)}(\mathbf{k}, z) = \frac{3}{7} [\Omega_m(z)]^{-1/143} \frac{i\mathbf{k}F(\mathbf{k}, z)}{k^2}. \quad (\text{E.56})$$

- (5)-4 Inverse Fourier transform to get the second order displacement field in real space, and move particle further by $\Psi^{(2)}(\mathbf{q}, z)$

$$\mathbf{x} = \mathbf{q} + \Psi^{(1)}(\mathbf{q}, z) + \Psi^{(2)}(\mathbf{q}, z). \quad (\text{E.57})$$

Finally, assign the velocity

$$\mathbf{v} = f_1(z)\mathcal{H}(z)\Psi^{(1)}(\mathbf{q}, z) + f_2(z)\mathcal{H}(z)\Psi^{(2)}(\mathbf{q}, z). \quad (\text{E.58})$$

Now, we have generated the second order Lagrangian initial condition.

E.5 Starting redshifts, initial condition generators, and convergence tests

How early in redshift should one start N -body simulations? Is the usual first-order Lagrangian perturbation theory, which is traditionally known as the Zel'dovich approximation (ZA) (e.g., Efstathiou et al., 1985), accurate enough for generating initial conditions

for N -body simulations? Will ZA converge as one increases the starting redshift? In this Appendix we show that ZA converges only very slowly: even $z_{start} = 400$ leaves an artificial suppression of power at the level of $\sim 1\%$ relative to a more accurate initial condition generated by the second-order Lagrangian perturbation theory (2LPT). This suppression is persistent at all the redshifts we have studied, $z = 1 - 6$. On the other hand, simulations starting from 2LPT initial conditions at $z_{start} = 300$ show convergence at $z < 6$, i.e., simulations starting at $z_{start} = 400$ give very similar results at $z < 6$.

Crocce et al. (2006) have shown that **ZA-generated initial conditions yield an artificial suppression of power spectrum measured from N -body simulations. They call this effect the “transient effect”.** The lower the starting redshift, z_{start} , is, the larger the artificial suppression of power becomes. This is a very important systematic error and must be taken into account when one is interested in making precision predictions for the power spectrum (Jeong & Komatsu, 2006) as well as for the mass function of dark matter halos (Lukić et al., 2007).

The transient effects occur when decaying modes are excited artificially by inaccurate initial condition generators. Although these decaying modes decay, they decay only slowly as a^{-1} when ZA is used to generate initial conditions (Crocce et al., 2006). For example, the transient effect reported in Crocce et al. (2006) is about 4% at $k = 1 \ h \text{ Mpc}^{-1}$ at $z = 3$ for a simulation starting at $z_{start} = 49$, and decays only slowly toward lower redshifts. The error at this level is unacceptable for testing precision calculations of the power spectrum (Jeong & Komatsu, 2006; McDonald, 2007; Matarrese & Pietroni, 2007).

Crocce et al. (2006) also show that simulations starting from 2LPT initial conditions, which are more accurate than ZA, still yield transient effects with the opposite sign: there is an artificial amplification of power. However, **an advantage of 2LPT is that the transient modes decay much more quickly than those from ZA.**

A natural question then arises: would ZA perform better as z_{start} is raised, and if so, how large should z_{start} be? The same question would apply to 2LPT as well. To answer these questions, we ran 10 simulations with 5 different starting redshifts, $z_{start} = 50, 100, 150, 300, \text{ and } 400$, whose initial conditions were generated from either ZA or 2LPT. We have used a publicly-available 2LPT initial condition generator developed by Roman Scoccimarro⁴ to generate initial conditions for N -body simulations. We have then used the **Gadget-2** code to evolve density fields. The cosmological parameters are exactly the same as those used in

⁴<http://cosmo.nyu.edu/roman/2LPT>

the main body of this paper. We chose to run simulations with a $(200 h^{-1} \text{ Mpc})^3$ box and 512^3 particles for these runs. Finally, we have used the same initial random seed for all of these runs to facilitate head-to-head comparison.

Figure E.1 shows $P(k)$ measured from 5 ZA runs with $z_{start} = 50, 100, 150, 300$, and 400 (bottom to top), divided by $P(k)$ from a 2LPT run with $z_{start} = 300$. In all cases the power is suppressed relative to 2LPT, and the suppression is persistent at all redshifts from $z = 6$ to 1. The amount of suppression decreases only slowly as we raise z_{start} .

Figure E.2 shows $P(k)$ measured from 5 2LPT runs with $z_{start} = 50, 100, 150, 300$, and 400 (top to bottom), divided by $P(k)$ from a 2LPT run with $z_{start} = 300$. The situation is reversed: the transient effects amplify the power, but the amount of amplification decays very quickly with z and z_{start} . We conclude that the transient effect is unimportant ($< 1\%$) at $z < 6$, if initial conditions are generated at $z_{start} = 300$ using 2LPT.

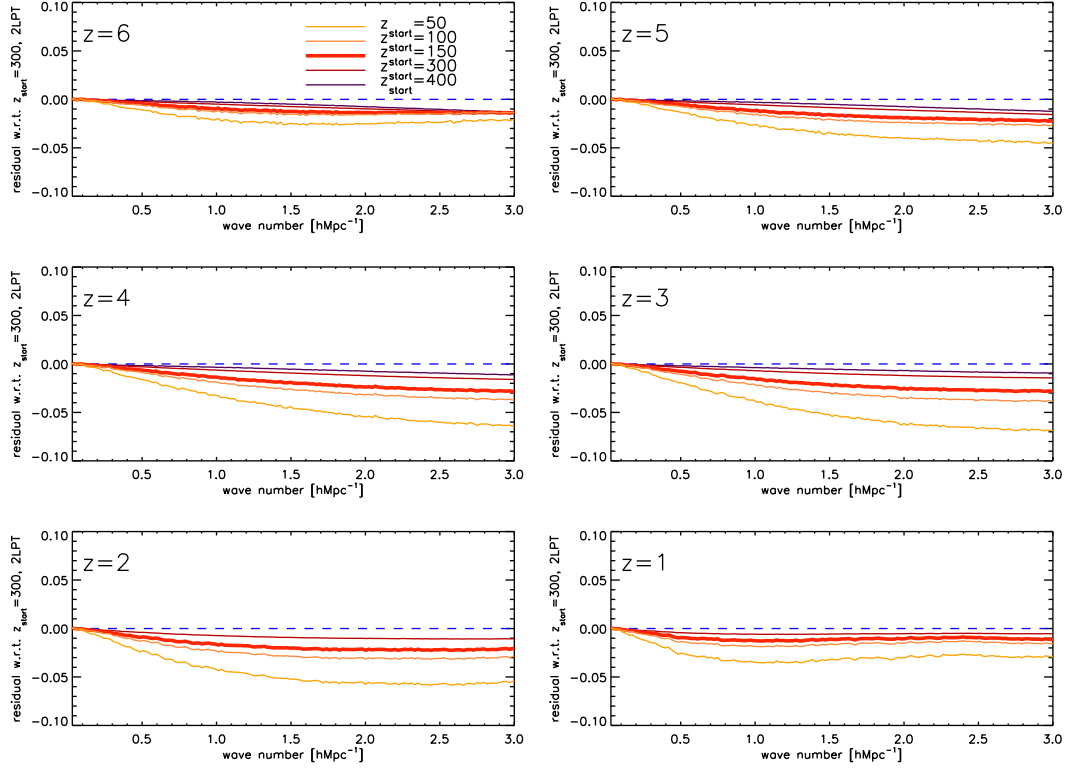


Figure E.1: Comparison between the power spectra calculated from 5 different ZA runs using different starting redshifts, $z_{start} = 50, 100, 150, 300$, and 400 (from bottom to top). The power spectra are divided by the one from 2LPT with $z_{start} = 300$, to facilitate comparison.

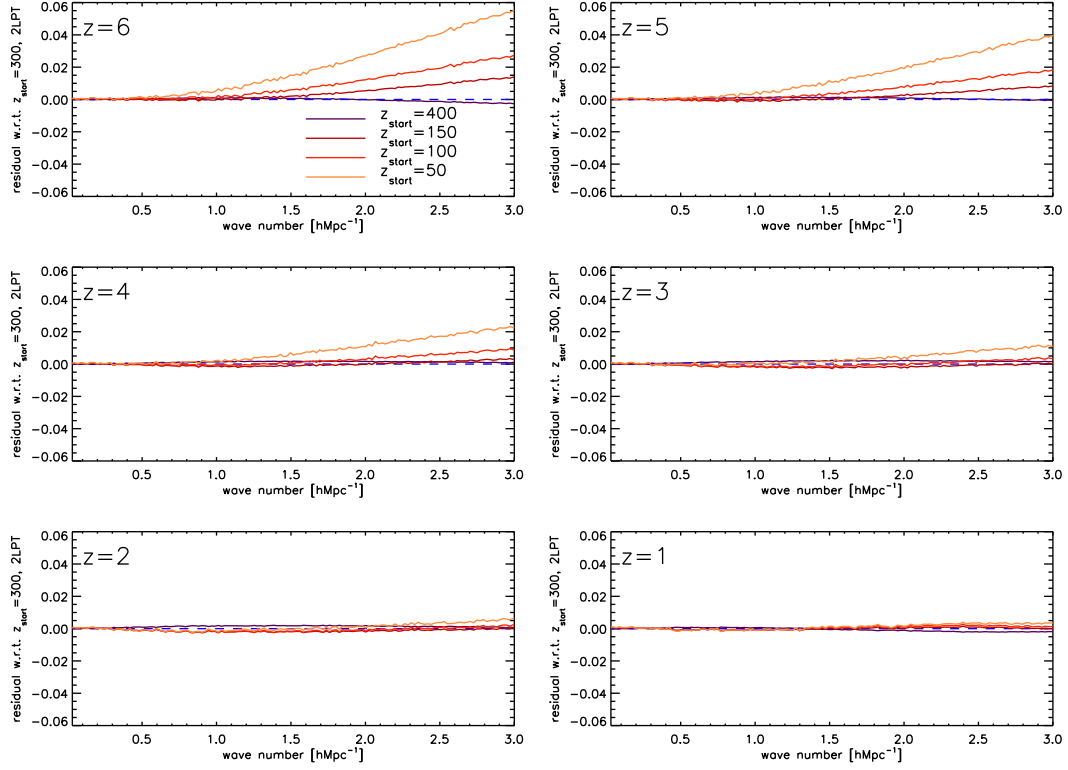


Figure E.2: Comparison between the power spectra calculated from 5 different 2LPT runs using different starting redshifts, $z_{\text{start}} = 50, 100, 150, 300$, and 400 (from bottom to top). The power spectra are divided by the one from 2LPT with $z_{\text{start}} = 300$, to facilitate comparison. Note that the power spectra with $z_{\text{start}} = 300$ and 400 agree very well, which suggests convergence at $z_{\text{start}} = 300$.